BEAM DESIGN FORMULAS WITH SHEAR AND MOMENT DIAGRAMS

DESIGN AID No. 6
BEAM FORMULAS WITH SHEAR AND MOMENT DIAGRAMS

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Figures 1 through 32 provide a series of shear and moment diagrams with accompanying formulas for design of beams under various static loading conditions.

Shear and moment diagrams and formulas are excerpted from the *Western Woods Use Book*, 4th edition, and are provided herein as a courtesy of Western Wood Products Association.

**Introduction**

**Notations Relative to “Shear and Moment Diagrams”**

- $E =$ modulus of elasticity, psi
- $I =$ moment of inertia, in.$^4$
- $L =$ span length of the bending member, ft.
- $l =$ span length of the bending member, in.
- $M =$ maximum bending moment, in.-lbs.
- $P =$ total concentrated load, lbs.
- $R =$ reaction load at bearing point, lbs.
- $V =$ shear force, lbs.
- $W =$ total uniform load, lbs.
- $w =$ load per unit length, lbs./in.
- $\Delta =$ deflection or deformation, in.
- $x =$ horizontal distance from reaction to point on beam, in.

**List of Figures**

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Simple Beam – Uniformly Distributed Load</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>Simple Beam – Uniform Load Partially Distributed</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>Simple Beam – Uniform Load Partially Distributed at One End</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>Simple Beam – Uniform Load Partially Distributed at Each End</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>Simple Beam – Load Increasing Uniformly to One End</td>
<td>6</td>
</tr>
<tr>
<td>6</td>
<td>Simple Beam – Load Increasing Uniformly to Center</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>Simple Beam – Concentrated Load at Center</td>
<td>7</td>
</tr>
<tr>
<td>8</td>
<td>Simple Beam – Concentrated Load at Any Point</td>
<td>7</td>
</tr>
<tr>
<td>9</td>
<td>Simple Beam – Two Equal Concentrated Loads Symmetrically Placed</td>
<td>8</td>
</tr>
<tr>
<td>10</td>
<td>Simple Beam – Two Equal Concentrated Loads Unsymmetrically Placed</td>
<td>8</td>
</tr>
<tr>
<td>11</td>
<td>Simple Beam – Two Unequal Concentrated Loads Unsymmetrically Placed</td>
<td>9</td>
</tr>
<tr>
<td>12</td>
<td>Cantilever Beam – Uniformly Distributed Load</td>
<td>9</td>
</tr>
<tr>
<td>13</td>
<td>Cantilever Beam – Concentrated Load at Free End</td>
<td>10</td>
</tr>
<tr>
<td>14</td>
<td>Cantilever Beam – Concentrated Load at Any Point</td>
<td>10</td>
</tr>
<tr>
<td>15</td>
<td>Beam Fixed at One End, Supported at Other – Uniformly Distributed Load</td>
<td>11</td>
</tr>
<tr>
<td>16</td>
<td>Beam Fixed at One End, Supported at Other – Concentrated Load at Center</td>
<td>11</td>
</tr>
<tr>
<td>17</td>
<td>Beam Fixed at One End, Supported at Other – Concentrated Load at Any Point</td>
<td>12</td>
</tr>
<tr>
<td>18</td>
<td>Beam Overhanging One Support – Uniformly Distributed Load</td>
<td>12</td>
</tr>
<tr>
<td>19</td>
<td>Beam Overhanging One Support – Uniformly Distributed Load on Overhang</td>
<td>13</td>
</tr>
<tr>
<td>20</td>
<td>Beam Overhanging One Support – Concentrated Load at End of Overhang</td>
<td>13</td>
</tr>
<tr>
<td>21</td>
<td>Beam Overhanging One Support – Concentrated Load at Any Point Between Supports</td>
<td>14</td>
</tr>
<tr>
<td>22</td>
<td>Beam Overhanging Both Supports – Unequal Overhangs – Uniformly Distributed Load</td>
<td>14</td>
</tr>
<tr>
<td>23</td>
<td>Beam Fixed at Both Ends – Uniformly Distributed Load</td>
<td>15</td>
</tr>
<tr>
<td>24</td>
<td>Beam Fixed at Both Ends – Concentrated Load at Center</td>
<td>15</td>
</tr>
<tr>
<td>25</td>
<td>Beam Fixed at Both Ends – Concentrated Load at Any Point</td>
<td>16</td>
</tr>
<tr>
<td>26</td>
<td>Continuous Beam – Two Equal Spans – Uniform Load on One Span</td>
<td>16</td>
</tr>
<tr>
<td>27</td>
<td>Continuous Beam – Two Equal Spans – Concentrated Load at Center of One Span</td>
<td>17</td>
</tr>
<tr>
<td>28</td>
<td>Continuous Beam – Two Equal Spans – Concentrated Load at Any Point</td>
<td>17</td>
</tr>
<tr>
<td>29</td>
<td>Continuous Beam – Two Equal Spans – Uniformly Distributed Load</td>
<td>18</td>
</tr>
<tr>
<td>30</td>
<td>Continuous Beam – Two Equal Spans – Two Equal Concentrated Loads Symmetrically Placed</td>
<td>18</td>
</tr>
<tr>
<td>31</td>
<td>Continuous Beam – Two Unequal Spans – Uniformly Distributed Load</td>
<td>19</td>
</tr>
<tr>
<td>32</td>
<td>Continuous Beam – Two Unequal Spans – Concentrated Load on Each Span Symmetrically Placed</td>
<td>19</td>
</tr>
</tbody>
</table>
Figure 1  Simple Beam – Uniformly Distributed Load

\[ R = V = \frac{w\ell}{2} \]
\[ V_x = w\left(\frac{\ell}{2} - x\right) \]
\[ M_{\text{max}} \text{ (at center)} = \frac{w\ell^2}{8} \]
\[ M_x = \frac{wx}{2} (\ell - x) \]
\[ \Delta_{\text{max}} \text{ (at center)} = \frac{5w\ell^4}{384EI} \]
\[ \Delta_x = \frac{wx}{24EI} (\ell^3 - 2\ell x^2 + x^3) \]

Figure 2  Simple Beam – Uniform Load Partially Distributed

\[ R_1 = V_1 \text{ (max when } a < c) = \frac{wb}{2\ell} \]
\[ R_2 = V_2 \text{ (max when } a > c) = \frac{wb}{2\ell} (2a + b) \]
\[ V_x \text{ (when } x > a \text{ and } < (a+b)) = R_1 - w(x-a) \]
\[ M_{\text{max}} \text{ (at } x = a + \frac{R_1}{w}) = R_1 \left(a + \frac{R_1}{2w}\right) \]
\[ M_x \text{ (when } x < a) = R_1x \]
\[ M_x \text{ (when } x > a \text{ and } < (a+b)) = R_1x - \frac{w}{2}(x-a)^2 \]
\[ M_x \text{ (when } x > (a+b)) = R_2(\ell-x) \]
Figure 3  Simple Beam – Uniform Load Partially Distributed at One End

\[ R_1 = V_1 \dots \dots \dots = \frac{wa}{2\ell} (2\ell - a) \]
\[ R_2 = V_2 \dots \dots \dots = \frac{wa^2}{2\ell} \]
\[ V_x \text{ (when } x < a) \dots \dots \dots = R_1 - wx \]
\[ M_{\text{max}} \left( \text{at } x = \frac{R_1}{w} \right) \dots \dots \dots = \frac{R_1^2}{2w} \]
\[ M_x \text{ (when } x < a) \dots \dots \dots = R_1 x - \frac{wx^2}{2} \]
\[ M_x \text{ (when } x > a) \dots \dots \dots = R_2 (\ell - x) \]
\[ \Delta_x \text{ (when } x < a) \dots \dots \dots = \frac{wx}{24\,E\,I\,l} \left( a^2 (2\ell - a)^2 - 2ax(2\ell - a) + a^2 x^2 \right) \]
\[ \Delta_x \text{ (when } x > a) \dots \dots \dots = \frac{wa^2(\ell - x)}{24\,E\,I\,l} (4x\ell - 2x^2 - a^2) \]

Figure 4  Simple Beam – Uniform Load Partially Distributed at Each End

\[ R_1 = V_1 \dots \dots \dots \dots = \frac{w_1 a (2\ell - a) + w_2 c^2}{2\ell} \]
\[ R_2 = V_2 \dots \dots \dots \dots = \frac{w_2 c (2\ell - c) + w_1 a^2}{2\ell} \]
\[ V_x \text{ (when } x < a) \dots \dots \dots = R_1 - w_1 x \]
\[ V_x \text{ (when } x > a \text{ and } < (a + b)) \dots \dots = R_1 - w_1 a \]
\[ V_x \text{ (when } x > (a + b)) \dots \dots \dots = R_2 - w_2 (\ell - x) \]
\[ M_{\text{max}} \left( \text{at } x = \frac{R_1}{w_1} \text{ when } R_1 < w_1 a \right) \dots \dots = \frac{R_1^2}{2w_1} \]
\[ M_{\text{max}} \left( \text{at } x = \ell - \frac{R_2}{w_2} \text{ when } R_2 < w_2 c \right) = \frac{R_2^2}{2w_2} \]
\[ M_x \text{ (when } x < a) \dots \dots \dots = R_1 x - \frac{w_1 x^2}{2} \]
\[ M_x \text{ (when } x > a \text{ and } < (a + b)) \dots \dots = R_1 x - \frac{w_1 a}{2} (2x - a) \]
\[ M_x \text{ (when } x > (a + b)) \dots \dots \dots = R_2 (\ell - x) - \frac{w_2 (\ell - x)^2}{2} \]
**Figure 5** Simple Beam – Load Increasing Uniformly to One End

\[ R_1 = V_1 \ldots \ldots \ldots \ldots = \frac{W}{3} \]
\[ R_2 = V_2 \ldots \ldots \ldots \ldots = \frac{2W}{3} \]
\[ V_x \ldots \ldots \ldots \ldots = \frac{W}{3} - \frac{Wx^2}{\ell^2} \]
\[ M_{\text{max}} \left( \text{at } x = \frac{\ell}{\sqrt{3}} = .5774\ell \right) \ldots \ldots = \frac{2W\ell}{9\sqrt{3}} = .1283W\ell \]
\[ M_x \ldots \ldots \ldots \ldots = \frac{Wx}{3\ell^2} (\ell^2 - x^2) \]
\[ \Delta_{\text{max}} \left( \text{at } x = \ell \sqrt{1 - \frac{8}{15} = .5193\ell} \right) = .01304 \frac{W\ell^3}{EI} \]
\[ \Delta_x \ldots \ldots \ldots \ldots = \frac{Wx}{180EI\ell^2} (3x^4 - 10\ell^2 x^2 + 7\ell^4) \]

**Figure 6** Simple Beam – Load Increasing Uniformly to Center

\[ R = V \ldots \ldots \ldots \ldots = \frac{W}{2} \]
\[ V_x \left( \text{when } x < \frac{\ell}{2} \right) \ldots \ldots = \frac{W}{2\ell^2} (\ell^2 - 4x^2) \]
\[ M_{\text{max}} \left( \text{at center} \right) \ldots \ldots = \frac{W\ell}{6} \]
\[ M_x \left( \text{when } x < \frac{\ell}{2} \right) \ldots \ldots = \frac{Wx}{2} \left( 1 - \frac{2x^2}{3\ell^2} \right) \]
\[ \Delta_{\text{max}} \left( \text{at center} \right) \ldots \ldots = \frac{W\ell^3}{60EI} \]
\[ \Delta_x \ldots \ldots \ldots \ldots = \frac{Wx}{480EI\ell^2} (5\ell^2 - 4x^2)^2 \]
Figure 7  Simple Beam – Concentrated Load at Center

\[ R = V \quad \ldots \ldots \ldots \ldots \quad = \frac{P}{2} \]
\[ M_{\text{max}} \text{ (at point of load)} \quad \ldots \ldots \ldots \quad = \frac{P\ell}{4} \]
\[ M_x \left( \text{when } x < \frac{\ell}{2} \right) \quad \ldots \ldots \ldots \quad = \frac{Px}{2} \]
\[ \Delta_{\text{max}} \text{ (at point of load)} \quad \ldots \ldots \ldots \quad = \frac{P\ell^3}{48EI} \]
\[ \Delta_x \left( \text{when } x < \frac{\ell}{2} \right) \quad \ldots \ldots \ldots \quad = \frac{Px}{48EI} \left(3\ell^2 - 4x^2\right) \]

Figure 8  Simple Beam – Concentrated Load at Any Point

\[ R_1 = V_1 \text{ (max when } a < b) \quad \ldots \ldots \ldots \quad = \frac{Pb}{\ell} \]
\[ R_2 = V_2 \text{ (max when } a > b) \quad \ldots \ldots \ldots \quad = \frac{Pa}{\ell} \]
\[ M_{\text{max}} \text{ (at point of load)} \quad \ldots \ldots \ldots \quad = \frac{Pab}{\ell} \]
\[ M_x \left( \text{when } x < a \right) \quad \ldots \ldots \ldots \quad = \frac{Pbx}{\ell} \]
\[ \Delta_{\text{max}} \left( \text{at } x = \sqrt{\frac{a(a + 2b)}{3}} \text{ when } a > b \right) \quad = \frac{Pab(a + 2b)\sqrt{3a(a + 2b)}}{27EI\ell} \]
\[ \Delta_x \text{ (at point of load)} \quad \ldots \ldots \ldots \quad = \frac{Pa^2b^2}{3EI\ell} \]
\[ \Delta_x \left( \text{when } x < a \right) \quad \ldots \ldots \ldots \quad = \frac{Pbx}{6EI\ell} \left(\ell^2 - b^2 - x^2\right) \]
\[ \Delta_x \left( \text{when } x > a \right) \quad \ldots \ldots \ldots \quad = \frac{Pa(\ell - x)}{6EI\ell} \left(2\ell x - x^3 - a^2\right) \]
Figure 9  Simple Beam – Two Equal Concentrated Loads Symmetrically Placed

\[ R = V \quad \ldots \ldots \ldots \ldots \ldots \ldots = P \]
\[ M_{\text{max}} \text{ (between loads)} \quad \ldots \ldots \ldots = Pa \]
\[ M_x \quad \text{ (when } x < a \quad \ldots \ldots \ldots = P \cdot x \]
\[ \Delta_{\text{max}} \text{ (at center)} \quad \ldots \ldots \ldots = \frac{Pa}{24EI} (3\ell^2 - 4a^2) \]
\[ \Delta_x \quad \text{ (when } x < a \quad \ldots \ldots \ldots = \frac{Px}{6EI} (3\ell a - 3a^2 - x^2) \]
\[ \Delta_x \quad \text{ (when } x > a \text{ and } < (\ell - a) \quad \ldots = \frac{Pa}{6EI} (3\ell x - 3x^2 - a^2) \]

Figure 10  Simple Beam – Two Equal Concentrated Loads Unsymmetrically Placed

\[ R_1 = V_1 \quad \text{ (max when } a < b \quad \ldots \ldots = \frac{P}{\ell} (\ell - a + b) \]
\[ R_2 = V_2 \quad \text{ (max when } a > b \quad \ldots \ldots = \frac{P}{\ell} (\ell - b + a) \]
\[ V_1 \quad \text{ (when } x > a \text{ and } < (\ell - b) \quad \ldots = \frac{P}{\ell} (b - a) \]
\[ M_1 \quad \text{ (max when } a > b \quad \ldots \ldots = R_1 a \]
\[ M_2 \quad \text{ (max when } a < b \quad \ldots \ldots = R_2 b \]
\[ M_x \quad \text{ (when } x < a \quad \ldots \ldots \ldots = R_1 x \]
\[ M_x \quad \text{ (when } x > a \text{ and } < (\ell - b) \quad \ldots = R_1 x - P(x - a) \]
**Figure 11** Simple Beam – Two Unequal Concentrated Loads Unsymmetrically Placed

\[ R_1 = V_1 = \frac{P_1(\ell - a) + P_2 b}{\ell} \]

\[ R_2 = V_2 = \frac{P_1 a + P_2(\ell - b)}{\ell} \]

\[ V_x \text{ when } x > a \text{ and } (\ell - b) < \ell \]

\[ M_1 \text{ (max when } R_1 < P_1) = R_1 a \]

\[ M_2 \text{ (max when } R_2 < P_2) = R_2 b \]

\[ M_x \text{ (when } x < a) = R_1 x \]

\[ M_x \text{ (when } x > a \text{ and } (\ell - b) < \ell) = R_1 x - P_1(x - a) \]

**Figure 12** Cantilever Beam – Uniformly Distributed Load

\[ R = V = w\ell \]

\[ V_x = wx \]

\[ M_{\text{max}} \text{ (at fixed end)} = \frac{w\ell^4}{2} \]

\[ M_x = \frac{wx^4}{2} \]

\[ \Delta_{\text{max}} \text{ (at free end)} = \frac{w\ell^4}{8EI} \]

\[ \Delta_x = \frac{w}{24EI} (x^4 - 4\ell^3 x + 3\ell^4) \]
**Figure 13  Cantilever Beam – Concentrated Load at Free End**

\[
R = V = P
\]

\[
M_{\text{max}} \text{ (at fixed end)} = P\ell
\]

\[
M_x = P\ell^3 / 3EI
\]

\[
\Delta_{\text{max}} \text{ (at free end)} = P\ell^3 / 3EI
\]

\[
\Delta_x = P \ell / 6EI (2\ell^3 - 3\ell^2 x + x^3)
\]

**Figure 14  Cantilever Beam – Concentrated Load at Any Point**

\[
R = V = P
\]

\[
M_{\text{max}} \text{ (at fixed end)} = Pb
\]

\[
M_x \text{ (when } x > a) = P(x - a)
\]

\[
\Delta_{\text{max}} \text{ (at free end)} = Pb^3 / 6EI (3\ell - b)
\]

\[
\Delta_x \text{ (at point of load)} = Pb^3 / 3EI
\]

\[
\Delta_x \text{ (when } x < a) = Pb^2 (3\ell - 3x - b) / 6EI
\]

\[
\Delta_x \text{ (when } x > a) = P(\ell - x)^2 / 6EI (3b - \ell + x)
\]
Figure 15  Beam Fixed at One End, Supported at Other – Uniformly Distributed Load

\[ R_1 = V_1 = \frac{3w\ell}{8} \]
\[ R_2 = V_2 = \frac{5w\ell}{8} \]
\[ V_x = R_1 - w\ell x \]
\[ M_{\text{max}} = \frac{w\ell^2}{8} \]
\[ M_1 \left( \text{at } x = \frac{3\ell}{8} \right) = \frac{9w\ell^2}{128} \]
\[ M_x = R_1 x - \frac{wx^2}{2} \]
\[ \Delta_{\text{max}} \left( \text{at } x = \frac{\ell}{16} \left( 1 + \sqrt{33} \right) = .4215 \ell \right) = \frac{w\ell^4}{185EI} \]
\[ \Delta_x = \frac{wx}{48EI} \left( \ell^3 - 3\ell x^2 + 2x^3 \right) \]

Figure 16  Beam Fixed at One End, Supported at Other – Concentrated Load at Center

\[ R_1 = V_1 = \frac{5P}{16} \]
\[ R_2 = V_2 = \frac{11P}{16} \]
\[ M_{\text{max}} \left( \text{at fixed end} \right) = \frac{3P\ell}{16} \]
\[ M_1 \left( \text{at point of load} \right) = \frac{5P\ell}{32} \]
\[ M_x \left( \text{when } x < \frac{\ell}{2} \right) = \frac{5Px}{16} \]
\[ M_x \left( \text{when } x > \frac{\ell}{2} \right) = P \left( \frac{\ell}{2} - \frac{11x}{16} \right) \]
\[ \Delta_{\text{max}} \left( \text{at } x = \ell \left( \frac{1}{5} \right) = .4472\ell \right) = \frac{P\ell^3}{48EI\sqrt{5}} = .009317\frac{P\ell^3}{EI} \]
\[ \Delta_x \left( \text{at point of load} \right) = 7\frac{P\ell^3}{768EI} \]
\[ \Delta_x \left( \text{when } x < \frac{\ell}{2} \right) = \frac{Px}{96EI} \left( 3\ell^2 - 5x^2 \right) \]
\[ \Delta_x \left( \text{when } x > \frac{\ell}{2} \right) = \frac{P}{96EI} (x - \ell)^2 (11x - 2\ell) \]
Figure 17  Beam Fixed at One End, Supported at Other – Concentrated Load at Any Point

\[ R_1 = V_1 = \frac{Pb^2}{2\ell} (a + 2\ell) \]

\[ R_2 = V_2 = \frac{Pa}{2\ell} (3\ell^2 - a^2) \]

\[ M_1 \text{ (at point of load)} = R_1a \]

\[ M_2 \text{ (at fixed end)} = \frac{Pab}{2\ell} (a + \ell) \]

\[ M_x \text{ (when } x < a) = R_1x \]

\[ M_x \text{ (when } x > a) = R_1x - P(x - a) \]

\[ \Delta_{\text{max}} \text{ (when } a < .414\ell\text{ at } x = \ell - \frac{\ell^2 + a^2}{3\ell^2 - a^2} = \frac{Pa}{3EI} \left( \frac{\ell^2 - a^2}{3\ell^2 - a^2} \right)^3 \]

\[ \Delta_{\text{max}} \text{ (when } a > .414\ell\text{ at } x = \ell - \frac{a}{2\ell + a} = \frac{Pab^2}{6EI} \left( \frac{a}{2\ell + a} \right)^2 \]

\[ \Delta_s \text{ (at point of load)} = \frac{Pa^2b^2}{12EI\ell^4} (3\ell + a) \]

\[ \Delta_s \text{ (when } x < a) = \frac{Pb^2x}{12EI\ell^4} (3a\ell^2 - 2\ell x^2 - ax^2) \]

\[ \Delta_s \text{ (when } x > a) = \frac{Pa}{12EI\ell^4} (\ell - x)^2(3\ell^2 - a^2 \ell - 2a^2\ell) \]

Figure 18  Beam Overhanging One Support – Uniformly Distributed Load

\[ R_1 = V_1 = \frac{w}{2\ell} (\ell^2 - a^2) \]

\[ R_2 = V_2 + V_1 = \frac{w}{2\ell} (\ell^2 + a^2) \]

\[ V_2 = wa \]

\[ V_3 = \frac{w}{2\ell} (\ell^2 + a^2) \]

\[ V_s \text{ (between supports)} = R_1 - wx \]

\[ V_s \text{ (for overhang)} = w(a - x_i) \]

\[ M_1 \left( \text{at } x = \ell - \frac{1 - \frac{a^2}{\ell^2}}{2} \right) = \frac{w}{8\ell^2} (\ell + a)^2(\ell - a)^2 \]

\[ M_2 \text{ (at } R_2) = \frac{wa^2}{2} \]

\[ M_s \text{ (between supports)} = \frac{wx}{2\ell} (\ell^2 - a^2 - x\ell) \]

\[ M_s \text{ (for overhang)} \]

\[ \Delta_s \text{ (between supports)} = \frac{wx}{24EI\ell} (\ell^4 - 2\ell^2 x^2 + \ell x^3 - 2a^2\ell^3 + 2a^3x^2) \]

\[ \Delta_s \text{ (for overhang)} = \frac{wx}{24EI\ell} (4a^2\ell - \ell^3 + 6a^2 x_i - 4ax_i^2 + x_i^3) \]
Figure 19  Beam Overhanging One Support – Uniformly Distributed Load on Overhang

\[ R_1 = V_1 = \frac{wa^2}{2\ell} \]
\[ R_2 = V_1 + V_2 = \frac{wa}{2\ell} (2\ell + a) \]
\[ V_2 = wa \]
\[ V_{x_1} \text{ (for overhang)} = wa(a - x_1) \]
\[ M_{\text{max}} \text{ (at } R_1) = \frac{wa^2}{2} \]
\[ M_x \text{ (between supports)} = \frac{wa^2 x}{2\ell} \]
\[ M_{x_1} \text{ (for overhang)} = \frac{w}{2} (a - x_1)^2 \]
\[ \Delta_{\text{max}} \left( \text{between supports at } x = \frac{\ell}{\sqrt{3}} \right) = \frac{wa^3 \ell}{18\sqrt{3}EI} = 0.03208 \frac{wa^3 \ell^4}{EI} \]
\[ \Delta_{\text{max}} \text{ (for overhang at } x_1 = a) = \frac{wa^3}{24EI} (4\ell + 3a) \]
\[ \Delta_x \text{ (between supports)} = \frac{wa^2 x}{12EI} (\ell^2 - x^2) \]
\[ \Delta_{x_1} \text{ (for overhang)} = \frac{wx_1}{24EI} (4a^2 \ell^2 + 6a^2 x_1 - 4ax_1^2 + x_1^3) \]

Figure 20  Beam Overhanging One Support – Concentrated Load at End of Overhang

\[ R_1 = V_1 = \frac{Pa}{\ell} \]
\[ R_2 = V_1 + V_2 = \frac{P}{\ell} (\ell + a) \]
\[ V_2 = P \]
\[ M_{\text{max}} \text{ (at } R_1) = \frac{Pa}{\ell} \]
\[ M_x \text{ (between supports)} = \frac{Pax}{\ell} \]
\[ M_{x_1} \text{ (for overhang)} = P(a - x_1) \]
\[ \Delta_{\text{max}} \left( \text{between supports at } x = \frac{\ell}{\sqrt{3}} \right) = \frac{Pa\ell^2}{9\sqrt{3}EI} = 0.06415 \frac{Pa\ell^2}{EI} \]
\[ \Delta_{\text{max}} \text{ (for overhang at } x_1 = a) = \frac{Pa^2}{3EI} (\ell + a) \]
\[ \Delta_x \text{ (between supports)} = \frac{Pax}{6EI} (\ell^2 - x^2) \]
\[ \Delta_{x_1} \text{ (for overhang)} = \frac{Px_1}{6EI} (2a\ell + 3ax_1 - x_1^2) \]
**Figure 21** Beam Overhanging One Support – Concentrated Load at Any Point Between Supports

\[ R_1 = V_1 \text{ (max when } a < b) = \frac{Pb}{\ell} \]

\[ R_2 = V_2 \text{ (max when } a > b) = \frac{Pa}{\ell} \]

\[ M_{\text{max}} \text{ (at point of load) } = \frac{Pab}{\ell} \]

\[ M_x \text{ (when } x < a) = \frac{Pbx}{\ell} \]

\[ \Delta_{\text{max}} \left( \text{at } x = \sqrt{\frac{a(a + 2b)}{3}} \text{ when } a > b \right) = \frac{Pab(a + 2b)\sqrt{3a(a + 2b)}}{27E\ell^3} \]

\[ \Delta_x \text{ (at point of load) } = \frac{Pa^2b^2}{3E\ell^3} \]

\[ \Delta_x \text{ (when } x < a) = \frac{Pbx(\ell^2 - b^2 - x^2)}{6E\ell^3} \]

\[ \Delta_x \text{ (when } x > a) = \frac{P\ell^2 - x^2 - a^2}{6E\ell^3} \]

\[ \Delta_{\text{max}} \text{ (at point of load) } = \frac{Pab\ell}{6E\ell^3} \]

**Figure 22** Beam Overhanging Both Supports – Unequal Overhangs – Uniformly Distributed Load

\[ R_1 = \frac{\omega \ell (\ell - 2c)}{2b} \]

\[ R_2 = \frac{\omega \ell (\ell - 2a)}{2b} \]

\[ V_1 = \omega a \]

\[ V_2 = R_1 - V_1 \]

\[ V_3 = R_2 - V_4 \]

\[ V_4 = \omega c \]

\[ V_{x_1} = V_1 - \omega x_1 \]

\[ V_x \text{ (when } x < \ell) = R_1 - \omega(a + x_1) \]

\[ V_m \text{ (when } a < c) = R_2 - \omega c \]

\[ M_1 = -\frac{\omega a^2}{2} \]

\[ M_2 = -\frac{\omega c^2}{2} \]

\[ M_3 = R_1 \left( \frac{R_1}{2w} - a \right) \]

\[ M_x \text{ (max when } x = \frac{R_1}{w} - a) = R_1 x - \frac{\omega(a + x)^2}{2} \]
Figure 23  Beam Fixed at Both Ends – Uniformly Distributed Load

\[ R = V = \frac{w\ell}{2} \]

\[ V_x = w\left(\frac{\ell}{2} - x\right) \]

\[ M_{\text{max}} \text{ (at ends)} = \frac{w\ell^2}{12} \]

\[ M_{i} \text{ (at center)} = \frac{w\ell^2}{24} \]

\[ M_x = \frac{w(6\ell x - \ell^2 - 6x^2)}{12} \]

\[ \Delta_{\text{max}} \text{ (at center)} = \frac{w\ell^4}{384EI} \]

\[ \Delta_x = \frac{wx^2}{24EI} (\ell - x)^2 \]

Figure 24  Beam Fixed at Both Ends – Concentrated Load at Center

\[ R = V = \frac{P}{2} \]

\[ M_{\text{max}} \text{ (at center and ends)} = \frac{P\ell}{8} \]

\[ M_x \left(\text{when } x < \frac{\ell}{2}\right) = \frac{P}{8}(4x - \ell) \]

\[ \Delta_{\text{max}} \text{ (at center)} = \frac{P\ell^3}{192EI} \]

\[ \Delta_x \left(\text{when } x < \frac{\ell}{2}\right) = \frac{Px^2}{48EI} (3\ell - 4x) \]
**Figure 25  Beam Fixed at Both Ends – Concentrated Load at Any Point**

\[ R_1 = V_1 \text{ (max when } a < b) \quad = \quad \frac{Pb^2}{\ell^3} (3a + b) \]

\[ R_2 = V_2 \text{ (max when } a > b) \quad = \quad \frac{Pa^2}{\ell^3} (a + 3b) \]

\[ M_1 \text{ (max when } a < b) \quad = \quad \frac{Pab^2}{\ell^4} \]

\[ M_2 \text{ (max when } a > b) \quad = \quad \frac{Pa^2b}{\ell^4} \]

\[ M_x \text{ (at point of load)} \quad = \quad \frac{2Pa^2b^2}{\ell^4} \]

\[ M_x \text{ (when } x < a) \quad = \quad R_1x - \frac{Pab^2}{\ell^2} \]

\[ \Delta_{max} \text{ (when } a > b \text{ at } x = \frac{2a\ell}{3a + b}) \quad = \quad \frac{2Pa^2b^2}{3EI(3a + b)^2} \]

\[ \Delta_x \text{ (at point of load)} \quad = \quad \frac{Pa^2b^3}{3EI^3} \]

\[ \Delta_x \text{ (when } x < a) \quad = \quad \frac{Pb^2x^2}{6EI^3} (3a\ell - 3ax - bx) \]

**Figure 26  Continuous Beam – Two Equal Spans – Uniform Load on One Span**

\[ R_1 = V_1 \quad = \quad \frac{7}{16} w\ell \]

\[ R_2 = V_2 + V_3 \quad = \quad \frac{5}{8} w\ell \]

\[ R_3 = V_3 = \quad = \quad -\frac{1}{16} w\ell \]

\[ V_1 = \quad = \quad \frac{9}{16} w\ell \]

\[ M_{max} \quad \text{ at } x = \frac{7}{16} \ell \quad = \quad \frac{49}{512} w\ell^2 \]

\[ M_1 \text{ (at support } R_2) \quad = \quad \frac{1}{16} w\ell^2 \]

\[ M_x \text{ (when } x < \ell) \quad = \quad \frac{wx}{16} (7\ell - 8x) \]
**Figure 27**  Continuous Beam – Two Equal Spans – Concentrated Load at Center of One Span

\[ R_1 = V_1 = \frac{13}{32} P \]
\[ R_2 = V_2 + V_1 = \frac{11}{16} P \]
\[ R_3 = V_3 = -\frac{3}{32} P \]
\[ V_2 = \frac{19}{32} P \]
\[ M_{\text{max}} \text{ (at point of load)} = \frac{13}{64} P\ell \]
\[ M_1 \text{ (at support } R_2) = \frac{3}{32} P\ell \]

**Figure 28**  Continuous Beam – Two Equal Spans – Concentrated Load at Any Point

\[ R_1 = V_1 = \frac{Pb}{4\ell^1} \left( 4\ell^1 - a(\ell + a) \right) \]
\[ R_2 = V_2 + V_1 = \frac{Pa}{2\ell^1} \left( 2\ell^1 + b(\ell + a) \right) \]
\[ R_3 = V_3 = -\frac{Pab}{4\ell^1} (\ell + a) \]
\[ V_2 = \frac{Pa}{4\ell^1} \left( 4\ell^1 + b(\ell + a) \right) \]
\[ M_{\text{max}} \text{ (at point of load)} = \frac{Pab}{4\ell^1} \left( 4\ell^1 - a(\ell + a) \right) \]
\[ M_1 \text{ (at support } R_2) = \frac{Pab}{4\ell^1} (\ell + a) \]
**Figure 29** Continuous Beam – Two Equal Spans – Uniformly Distributed Load

\[ R_1 = V_1 = R_3 = \frac{3w\ell}{8} \]
\[ R_2 = \frac{10w\ell}{8} \]
\[ V_2 = \frac{5w\ell}{8} \]
\[ M_1 = \frac{w\ell^2}{8} \]
\[ M_2 = \frac{9w\ell^2}{128} \]
\[ \Delta_{\text{max}} = 0.4215\ell \text{ (approx. from } R_1 \text{ and } R_3) \]

**Figure 30** Continuous Beam – Two Equal Spans – Two Equal Concentrated Loads Symmetrically Placed

\[ R_1 = V_1 = R_3 = \frac{5P}{16} \]
\[ R_2 = 2V_2 = \frac{11P}{8} \]
\[ V_2 = P - R_1 = \frac{11P}{16} \]
\[ V_{\text{max}} = V_2 \]
\[ M_1 = -\frac{3P\ell}{16} \]
\[ M_2 = \frac{5P\ell}{32} \]
\[ M_4 \text{ (when } x < a) = R_1x \]
**Figure 31** Continuous Beam – Two Unequal Spans – Uniformly Distributed Load

\[ R_1 = \frac{M_1}{\ell_1} + \frac{w\ell_1}{2} \]
\[ R_2 = w\ell_1 + w\ell_2 - R_1 - R_3 \]
\[ R_3 = V_4 = \frac{M_1}{\ell_2} + \frac{w\ell_2}{2} \]
\[ V_1 = R_1 \]
\[ V_2 = w\ell_1 - R_1 \]
\[ V_3 = w\ell_2 - R_3 \]
\[ V_4 = R_1 \]
\[ M_1 = \frac{w\ell_1^3 + w\ell_2^3}{8(\ell_1 + \ell_2)} \]
\[ M_{x_1} \left( \text{when } x_1 = \frac{R_1}{w} \right) = R_1 x_1 - \frac{wx_1^2}{2} \]
\[ M_{x_2} \left( \text{when } x_2 = \frac{R_3}{w} \right) = R_3 x_2 - \frac{wx_2^2}{2} \]

**Figure 32** Continuous Beam – Two Unequal Spans – Concentrated Load on Each Span Symmetrically Placed

\[ R_1 = \frac{M_1}{\ell_1} + \frac{P_1}{2} \]
\[ R_2 = P_1 + P_2 - R_1 - R_3 \]
\[ R_3 = \frac{M_1}{\ell_2} + \frac{P_2}{2} \]
\[ V_1 = R_1 \]
\[ V_2 = P_1 - R_1 \]
\[ V_3 = P_2 - R_3 \]
\[ V_4 = R_1 \]
\[ M_1 = -\frac{3}{16} \left( \frac{P_1\ell_1^2 + P_2\ell_2^2}{\ell_1 + \ell_2} \right) \]
\[ M_{x_1} = R_1 a \]
\[ M_{x_2} = R_1 b \]