CHAPTER SEVEN

DEFLECTION OF SYMMETRIC BEAMS

Learning Objective

1. Learn to formulate and solve the boundary-value problem for the deflection of a beam at any point.

Greg Louganis, the American often considered the greatest diver of all time, has won four Olympic gold medals, one silver medal, and five world championship gold medals. He won both the springboard and platform diving competitions in the 1984 and 1988 Olympic games. In his incredible execution, Louganis and all divers (Figure 7.1*a*) makes use of the behavior of the diving board. The flexibility of the springboard, for example, depends on its thin aluminum design, with the roller support adjusted to give just the right unsupported length. In contrast, a bridge (Figure 7.1*b*) must be stiff enough so that it does not vibrate too much as the traffic goes over it. The stiffness in a bridge is obtained by using steel girders with a high area moment of inertias and by adjusting the distance between the supports. In each case, to account for the right amount of flexibility or stiffness in beam design, we need to determine the beam deflection, which is the topic of this chapter

(a) (b)

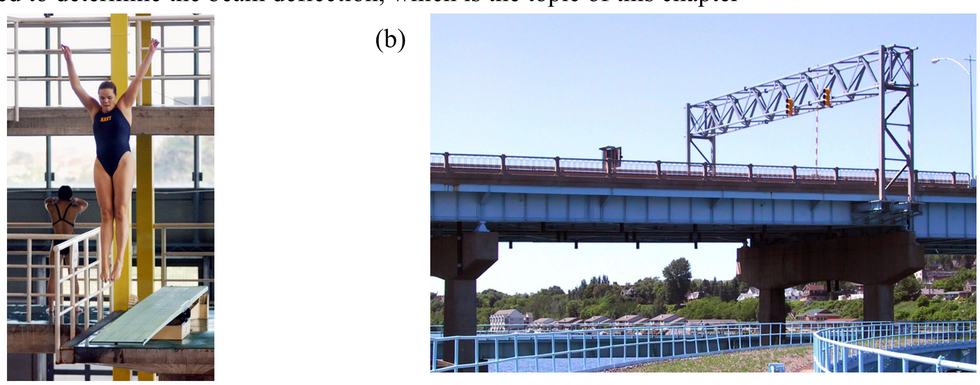


Figure 7.1 Examples of beam: (a) flexibility of diving board; and (b) stiffness of steel girders.

We can obtain the deflection of a beam by integrating either a second-order or a fourth-order differential equation. The differential equation, together with all the conditions necessary to solve for the integration constants, is called a **boundary-value problem**. The solution of the boundary-value problem gives the deflection at any location x along the length of the beam.

7.1 SECOND-ORDER BOUNDARY-VALUE PROBLEM

Chapter 6 considered the symmetric bending of beams. We found that if we can find the deflection in the y direction of one point on the cross section, then we know the deflection of all points on the cross section. In other words, the deflection at a cross section is independent of the y and z coordinates. However, the deflection can be a function of x, as shown in Figure 7.2. The deflected curve represented by v(x) is called the **elastic curve.**

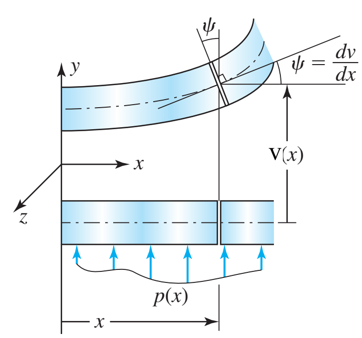


Figure 7.2 Beam deflection.

The deflection function v(x) can be found by integrating Equation (6.11) twice, provided we can find the internal moment as a function of x, as we did in Section 6.3. Equation (6.11), this second-order differential equation is rewritten for convenience:

$$\boldsymbol{M}_z = EI_{zz}\frac{d^2\mathrm{v}}{dx^2} \tag{7.1}$$

The two integration constants generated from Equation (7.12.a) are determined from boundary conditions, as discussed next, in Section 7.1.1.

As one moves across the beam, the applied load may change, resulting in different functions of x that represent the internal moment $\boldsymbol{M}_z$. In such cases there are as many differential equations as there are functions representing the moment $\boldsymbol{M}_z$. Each additional differential equation generates additional integration constants. These additional integration constants are determined from continuity (compatibility) equations, obtained by considering the point where the functional representation of the moment changes character. The continuity conditions will be discussed in Section 7.1.2. The mathematical statement listing all the differential equations and all the conditions necessary for solving for v(x) is called the **boundary-value problem** for the beam deflection.

7.1.1 Boundary Conditions

The integration of Equation (7.1) will result in v and $d\mathrm{v}/dx$. Thus, we are seeking conditions on v or $d\mathrm{v}/dx$. Figure 7.3 shows three types of support and the associated boundary conditions.

Note that for a second-order differential equation we need two boundary conditions. If on one end there is only one boundary condition, as in Figure 7.3*b* or *c*, then the remaining boundary condition must come from another location. Doubts about a boundary condition at a support can often be resolved by drawing an approximate deformed shape of the beam.

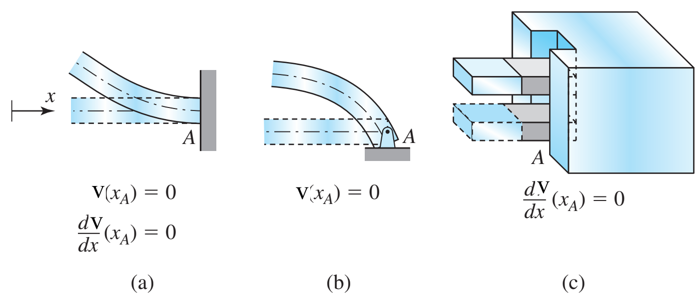


Figure 7.3 Boundary conditions for second-order differential equations. (*a*) Built-in end. (*b*) Simple support. (*c*) Smooth slot.

7.1.2 Continuity Conditions

Suppose that because of change in the applied loading, the internal moment $\boldsymbol{M}_z$ in a beam is represented by one function to the left of x_j and another function to the right of x_j. Then there are two second-order differential equations, and integration will produce two different displacement functions, one for each side of x_j, together, these will contain a total of four integration constants. Two of these four integration constants can be determined from the boundary conditions, as discussed in Section 7.1.1. The remaining two constants will have to be determined from conditions at x_j. Figure 7.4 shows that a discontinuous displacement at x_j implies a broken beam, and a discontinuous slope at x_j implies that a beam is kinked at x_j.

Assuming that the beam neither breaks nor kinks, then the displacement functions must satisfy the following conditions:

$$\mathrm{v}_1(x_j) = \mathrm{v}_2(x_j) \tag{7.2.a}$$

$$\frac{d\mathrm{v}_1}{dx}(x_j) = \frac{d\mathrm{v}_2}{dx}(x_j) \tag{7.2.b}$$

where v_1 and v_2 are the displacement functions to the left and right of x_j. The conditions given by Equations (7.2) are the **continuity conditions**, also known as compatibility conditions or matching conditions.

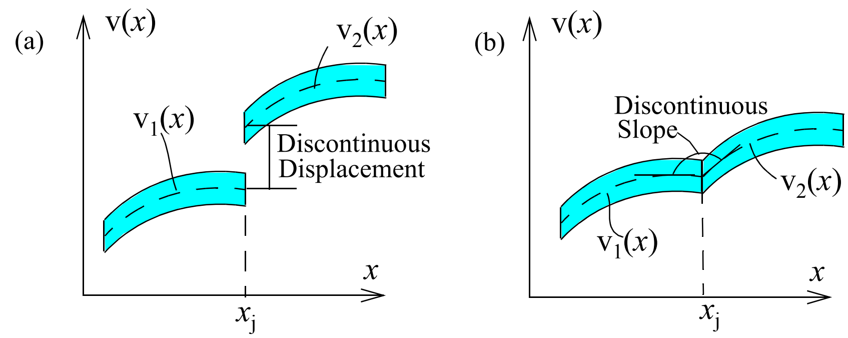


Figure 7.4 (*a*) Broken beam. (*b*) Kinked beam.

- Example 7.1 demonstrates the formulation and solution of a boundary-value problem with one second-order differential equation and the associated boundary conditions.
- Example 7.2 demonstrates the formulation and solution of a boundary-value problem with two second-order differential equations, the associated boundary conditions, and the continuity conditions.
- Example 7.3 demonstrates the formulation only of a boundary-value problem with multiple second-order differential equations, the associated boundary conditions, and the continuity conditions.
- Example 7.4 demonstrates the formulation and solution of a boundary-value problem with variable area moment of inertia, that is, I_{zz} is a function of x.

EXAMPLE 7.1

A beam has a linearly varying distributed load, as shown in Figure 7.5. Determine: (a) The equation of the elastic curve in terms of E, I, w, L, and x. (b) The maximum intensity of the distributed load if the maximum deflection is to be limited to 20 mm. Use $E = 200$ GPa, $I = 600\ (10^6)\ \text{mm}^4$, and $L = 8$ m.

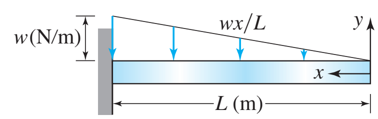


Figure 7.5 Beam and loading in Example 7.1.

PLAN

(a) We can make an imaginary cut at an arbitrary location x and draw the free-body diagram. Using equilibrium equations, the moment $\boldsymbol{M}_z$ can be written as a function of x. By integrating Equation (7.1) and using the boundary conditions that deflection and slope at $x = L$ are zero, we can find v(x). (b) The maximum deflection for this problem will occur at the free end and can be found by substituting $x = 0$ in the v(x) expression. By requiring that $|v_{max}| \le 0.02\ \text{m}$, we can find w_{max}.

SOLUTION

(a) Figure 7.6 shows the free-body diagram of the right part after making an imaginary cut at some location x. Internal moment and shear forces are drawn according to the sign convention discussed in Section 6.2.6. The distributed force is replaced by an equivalent force, and the internal moment is found by equilibrium of moment about point O.

$$\boldsymbol{M}_z = -\frac{1}{2}\frac{wx^2}{L}\left(\frac{x}{3}\right) = -\frac{1}{6}\frac{wx^3}{L} \tag{E1}$$

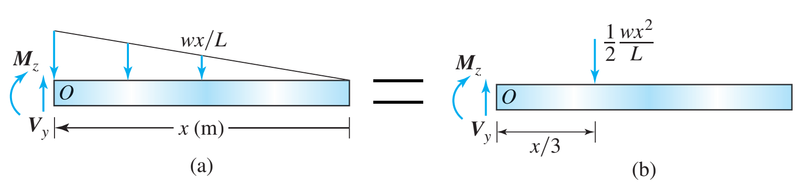


Figure 7.6 Free-body diagram in Example 7.1. (a) Imaginary cut on original beam. (b) Statically equivalent diagram.

We substitute Equation (E1) into Equation (7.1) and note the zero slope and deflection at the built-in end. The boundary-value problem can then be stated as follows:

- **Differential equation:**

$$EI_{zz}\frac{d^2v}{dx^2} = -\frac{1}{6}\frac{wx^3}{L} \tag{E2}$$

- **Boundary conditions:**

$$v(L) = 0 \tag{E3}$$

$$\frac{dv}{dx}(L) = 0 \tag{E4}$$

Equation (E2) can be integrated to obtain

$$EI_{zz}\frac{dv}{dx} = -\frac{1}{24}\frac{wx^4}{L} + c_1 \tag{E5}$$

Substituting $x = L$ in Equation (E5) and using Equation (E4) gives the constant c_1:

$$-\frac{1}{24}\frac{wL^4}{L} + c_1 = 0 \qquad \text{or} \qquad c_1 = \frac{wL^3}{24} \tag{E6}$$

Substituting Equation (E6) into Equation (E5) we obtain

$$EI_{zz}\frac{dv}{dx} = -\frac{1}{24}\frac{wx^4}{L} + \frac{wL^3}{24} \tag{E7}$$

Equation (E7) can be integrated to obtain

$$EI_{zz}v = -\frac{1}{120}\frac{wx^5}{L} + \frac{wL^3}{24}x + c_2 \tag{E8}$$

Substituting $x = L$ in Equation (E8) and using Equation (E3) gives the constant c_2:

$$-\frac{1}{120}\frac{wL^5}{L} + \frac{wL^3}{24}L + c_2 = 0 \qquad \text{or} \qquad c_2 = -\frac{wL^4}{30} \tag{E9}$$

The deflection expression can be obtained by substituting Equation (E9) into Equation (E8) and simplifying.

ANS. $v(x) = -\frac{w}{120EI_{zz}L}(x^5 - 5L^4x + 4L^5)$

Dimension check: We note that all terms in the parentheses have the dimension of length to the power of five, that is, $O(L^5)$. Thus the answer is dimensionally homogeneous. But we can also check whether the left-hand side and any one term of the right-hand side has the same dimension,

$$w \to O\left(\frac{F}{L}\right) \qquad x \to O(L) \qquad E \to O\left(\frac{F}{L^2}\right) \qquad I_{zz} \to O(L^4) \qquad v \to O(L) \qquad \frac{wx^5}{EI_{zz}L} \to O\left(\frac{(F/L)L^5}{(F/L^2)O(L^4)L}\right) \to O(L) \to \text{checks}$$

(b) By inspection it can be seen that the maximum deflection for this problem will occur at the free end. Substituting $x = 0$ in the deflection expression, we obtain $v_{max} = -wL^4/30EI_{zz}$. The minus sign indicates that the deflection is in the negative y direction, as expected. Substituting the given values of the variables and requiring that the magnitude of the deflection be less than 0.02 m, we obtain

$$|v_{max}| = \frac{w_{max}L^4}{30EI_{zz}} = \frac{w_{max}(8\text{ m})^4}{30[200(10^9)\text{ N/m}^2)][600(10^{-6})\text{ m}^4]} \le 0.02\text{ m} \qquad \text{or} \qquad w_{max} \le 17.58(10^3)\text{ N/m} \tag{E10}$$

ANS. $w_{max} = 17.5\text{ kN/m}$

COMMENTS

1. From calculus we know that the maximum of a function occurs at the point where the slope of the function is zero. But the slope at $x = L$, where the deflection is maximum, is not zero. This is because $v(x)$ is a **monotonic function**— that is, a continuously increasing (or decreasing) function. For monotonic functions the maximum (or minimum) always occurs at the end of the interval. We intuitively recognized the function's monotonic character when we stated that the maximum deflection occurs at the free end.
2. If the dimension check showed that some term did not have the proper dimension, then we would backtrack, check each equation for dimensional homogeneity, and identify the error.

EXAMPLE 7.2

For the beam and loading shown in Figure 7.7, determine: (a) the equation of the elastic curve in terms of *E*, *I*, *L*, *P*, and *x*; (b) the maximum deflection in the beam.

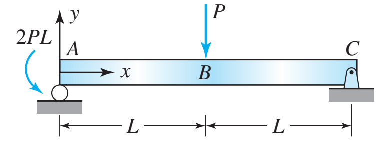


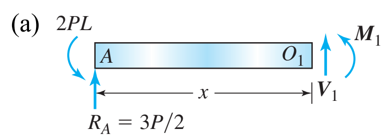
Figure 7.7 Beam and loading in Example 7.2.

PLAN

(a) The internal moment due to the load *P* at *B* will be represented by different functions in *AB* and *BC*, which can be found by making imaginary cuts and drawing free-body diagrams. We can write the two differential equations using Equation (7.1), the two boundary conditions of zero deflection at *A* and *C*, and the two continuity conditions at *B*. The boundary-value problem can be solved to obtain the elastic curve. (b) In each section we can set the slope to zero and find the roots of the equation that will give the location of zero slope. We can substitute the location values in the elastic curve equation derived in part (a) to determine the maximum deflection in the beam.

SOLUTION

(a) The free-body diagram of the entire beam can be drawn, and the reaction at *A* found as $R_A = 3P/2$ upward, and the reaction at *C* found as $R_C = P/2$ downward. Figure 7.8 shows the free body diagrams after imaginary cuts have been made and then internal shear force and bending moment drawn according to our sign convention.



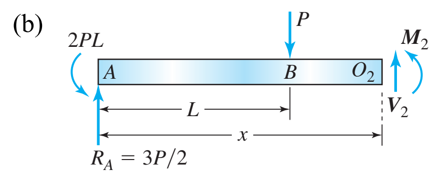


Figure 7.8 Free body diagrams in Example 7.2 after imaginary cut in (a) *AB* (b) *BC*.

By equilibrium of moments in Figure 7.8*a* and *b* we obtain the internal moments

$$\boldsymbol{M}_1 + 2PL - R_A x = 0 \qquad \text{or} \qquad \boldsymbol{M}_1 = \frac{3}{2}Px - 2PL \tag{E1}$$

$$\boldsymbol{M}_2 + 2PL - R_A x + P(x-L) = 0 \qquad \text{or} \qquad \boldsymbol{M}_2 = \frac{3}{2}Px - 2PL - P(x-L) \tag{E2}$$

Check: The internal moment must be continuous at *B*, since there is no external point moment at *B*. Substituting $x = L$ in Equations (E1) and (E2), we find $\boldsymbol{M}_1 = \boldsymbol{M}_2$ at $x = L$.

The boundary-value problem can be stated using Equation (7.1), (E1), and (E2), the zero deflection at points *A* and *C*, and the continuity conditions at *B* as follows:

- **Differential equations:**

$$EI_{zz}\frac{d^2 v_1}{dx^2} = \frac{3}{2}Px - 2PL, \qquad 0 \le x < L \tag{E3}$$

$$EI_{zz}\frac{d^2 v_2}{dx^2} = \frac{3}{2}Px - 2PL - P(x-L), \qquad L \le x < 2L \tag{E4}$$

- **Boundary conditions:**

$$v_1(0) = 0 \tag{E5}$$

$$v_2(2L) = 0 \tag{E6}$$

- **Continuity conditions:**

$$v_1(L) = v_2(L) \tag{E7}$$

$$\frac{dv_1}{dx}(L) = \frac{dv_2}{dx}(L) \tag{E8}$$

Integrating Equations (E3) and (E4) we obtain

$$EI_{zz}\frac{dv_1}{dx} = \frac{3}{4}Px^2 - 2PLx + c_1 \tag{E9}$$

$$EI_{zz}\frac{dv_2}{dx} = \frac{3}{4}Px^2 - 2PLx - \frac{P}{2}(x-L)^2 + c_2 \tag{E10}$$

Substituting $x = L$ in Equations (E9) and (E10) and using Equation (E8), we obtain

$$\frac{3}{4}PL^2 - 2PL^2 + c_1 = \frac{3}{4}PL^2 - 2PL^2 - 0 + c_2 \qquad \text{or} \qquad c_1 = c_2 \tag{E11}$$

Substituting Equation (E11) into Equation (E10) and integrating Equations (E9) and (E10), we obtain

$$EI_{zz}\mathrm{v}_1 = \frac{1}{4}Px^3 - PLx^2 + c_1x + c_3 \tag{E12}$$

$$EI_{zz}\mathrm{v}_2 = \frac{1}{4}Px^3 - PLx^2 - \frac{P}{6}(x-L)^3 + c_1x + c_4 \tag{E13}$$

Substituting $x = L$ in Equations (E12) and (E13) and using Equation (E7), we obtain

$$\frac{1}{4}PL^3 - PL^3 + c_1L + c_3 = \frac{1}{4}PL^3 - PL^3 - 0 + c_1L + c_4 \qquad \text{or} \qquad c_3 = c_4 \tag{E14}$$

Substituting $x = 0$ in Equation (E12) and using Equation (E5), we obtain

$$c_3 = 0 \tag{E15}$$

From Equation (E14),

$$c_4 = 0 \tag{E16}$$

Substituting $x = 2L$ and Equation (E16) into Equation (E13) and using Equation (E6), we obtain

$$\frac{1}{4}P(2L)^3 - PL(2L)^2 - \frac{P}{6}(L)^3 + c_1(2L) = 0 \qquad \text{or} \qquad c_1 = \frac{13}{12}PL^2 \tag{E17}$$

Substituting Equations (E15), (E16), and (E17) into Equations (E12) and (E13) and simplifying, we obtain the answer:

$$\textbf{ANS. } \mathrm{v}_1(x) = \frac{P}{12EI_{zz}}(3x^3 - 12Lx^2 + 13L^2x) \qquad 0 \le x < L \tag{E18}$$

$$\textbf{ANS. } \mathrm{v}_2(x) = \frac{P}{12EI_{zz}}[3x^3 - 12Lx^2 + 13L^2x - 2(x-L)^3] \qquad L \le x < 2L \tag{E19}$$

Dimension check: All terms in parentheses are dimensionally homogeneous, as all have the dimensions of length cubed. But we can also check whether the left-hand side and any one term of the right-hand side have the same dimension:

$$P \to O(F) \qquad x \to O(L) \qquad E \to O\left(\frac{F}{L^2}\right) \qquad I_{zz} \to O(L^4) \qquad \mathrm{v} \to O(L) \qquad \frac{Px^3}{EI_{zz}} \to O\left(\frac{FL^3}{(F/L^2)L^4}\right) \to O(L) \to \text{checks}$$

(b) Let $d\mathrm{v}_1/dx$ be zero at $x = x_1$. Differentiating Equation (E18), we obtain

$$\frac{P}{12EI_{zz}}(9x_1^2 - 24Lx_1 + 13L^2) = 0 \qquad \text{or} \qquad 9x_1^2 - 24Lx_1 + 13L^2 = 0 \tag{E20}$$

The roots of the quadratic equation are $x_1 = 1.91L$ and $x_1 = 0.756L.$. The admissible root is $x_1 = 0.756L.$, since Equation (E18), and hence Equation (E20), are valid only in the range from 0 to L. Substituting this root into Equation (E18), we obtain

$$\mathrm{v}_1(0.756L) = \frac{P}{12EI_{zz}}(3 \times 0.756L^3 - 12L \times 0.756L^2 + 13L^2 \times 0.756L) = \frac{0.355PL^3}{EI_{zz}} \tag{E21}$$

To find the maximum deflection in BC, assume $d\mathrm{v}_2/dx$ to be zero at $x = x_2$. Differentiating Equation (E19) we obtain

$$\frac{P}{12EI_{zz}}[9x_2^2 - 24Lx_2 + 13L^2 - 6(x_2 - L)^2] = 0 \qquad \text{or} \qquad 3x_2^2 - 12Lx_2 + 7L^2 = 0 \tag{E22}$$

The roots of the quadratic equations in Equation (E22) are $x_2 = 0.709L$ and $x_2 = 3.29L$. Both roots are outside the range of L to $2L$ and hence are inadmissible. Thus in this problem the slope is zero only at $0.756L$, and the maximum deflection is given by Equation (E21).

$$\textbf{ANS.} \qquad \mathrm{v}_{\max} = \frac{0.355PL^3}{EI_{zz}}$$

COMMENT

1. When we made the imaginary cut in BC, we took the left part for drawing the free-body diagram. Had we taken the right part, we would have obtained the moment expression $\boldsymbol{M}_2 = (Px/2) - PL$, which is the simplified form of Equation (E21). We can start with this moment expression and obtain our results from integration and the conditions as shown. The values of the integration constants will be different, and there will be slightly more algebra, but the final result will be the same. The form of the moment expression used in the example made use of the observation that the continuity conditions are at $x = L$ and the terms in powers of $(x - L)$ will be zero. This form results in less algebra and simplified relations for the constants, as given by Equations (E11) and (E14).

EXAMPLE 7.3

Write the boundary-value problem for solving the deflection at any point x of the beam shown in Figure 7.9. Do not integrate or solve.

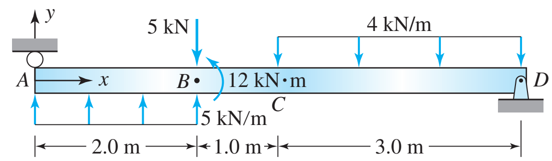


Figure 7.9 Beam and loading in Example 7.3.

PLAN

The moment expressions in each interval were found in Example 6.10. The differential equations can be written by substituting these moment expressions into Equation (7.1). We can also write the zero-deflection conditions at points A and D and the continuity conditions at points B and C to complete the boundary-value problem statement.

SOLUTION

From the free body diagram of the entire beam the reactions at A and D in Example 6.10 were found to be $R_A = 0$ and $R_D = 7\text{ kN}$. Figure 7.10 shows the free body diagrams used in Example 6.10 to obtain the internal moments

$$\boldsymbol{M}_1 = \left(\frac{5}{2}x^2\right)\text{ kN}\cdot\text{m} \qquad 0 \le x < 2\text{ m} \tag{E1}$$

$$\boldsymbol{M}_2 = (5x - 12)\text{ kN}\cdot\text{m} \qquad 2\text{ m} < x < 3\text{ m} \tag{E2}$$

$$\boldsymbol{M}_3 = (-2x^2 + 17x - 30)\text{ kN}\cdot\text{m} \qquad 3\text{ m} < x \le 6\text{ m} \tag{E3}$$

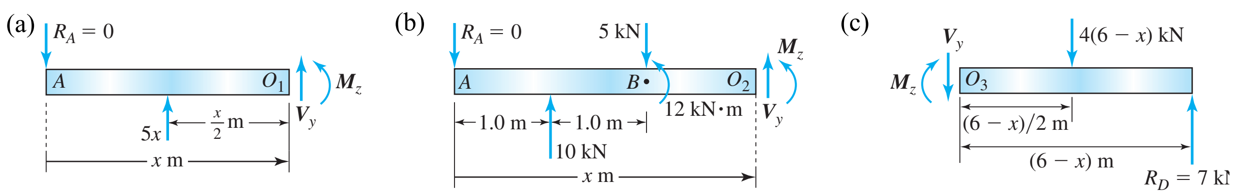


Figure 7.10 Free body diagrams in Example 7.3 after imaginary cut in (a) AB (b) BC (c) CD.

The boundary value problem can be written as described below.

- **Differential equations:**

$$EI_{zz}\frac{d^2v_1}{dx^2} = \left(\frac{5}{2}x^2\right)\text{ kN}\cdot\text{m} \qquad 0 \le x < 2\text{ m} \tag{E4}$$

$$EI_{zz}\frac{d^2v_2}{dx^2} = (5x - 12)\text{ kN}\cdot\text{m} \qquad 2\text{ m} < x < 3\text{ m} \tag{E5}$$

$$EI_{zz}\frac{d^2v_3}{dx^2} = (-2x^2 + 17x - 30)\text{ kN}\cdot\text{m} \qquad 3\text{ m} < x \le 6\text{ m} \tag{E6}$$

- **Boundary conditions:**

$$v_1(0) = 0 \tag{E7}$$

$$v_3(6) = 0 \tag{E8}$$

- **Continuity conditions:**

$$v_1(2) = v_2(2) \tag{E9}$$

$$\frac{dv_1}{dx}(2) = \frac{dv_2}{dx}(2) \tag{E10}$$

$$v_2(3) = v_3(3) \tag{E11}$$

$$\frac{dv_2}{dx}(3) = \frac{dv_3}{dx}(3) \tag{E12}$$

COMMENTS

1. Equations (E4), (E5), and (E6) are three differential equations of order 2. Integrating these three differential equations would result in six integration constants. We have two boundary conditions and four continuity conditions. A properly formulated boundary-value problems will always have *exactly* the right number of conditions needed to solve a problem.
2. In Example 7.3 there were two differential equations and the resulting algebra was tedious. This example has three differential equations, which will make the algebra even more tedious. Fortunately there is a method, discussed in Section 7.4*, which reduces the algebra. This *discontinuity method* introduces functions that will let us write all three differential equations as a single equation and implicitly satisfy the continuity conditions during integration.

EXAMPLE 7.4

A cantilever beam with variable width $b(x)$ is shown in Figure 7.11. Determine the maximum deflection in terms of P, b_L, t, L, and E.

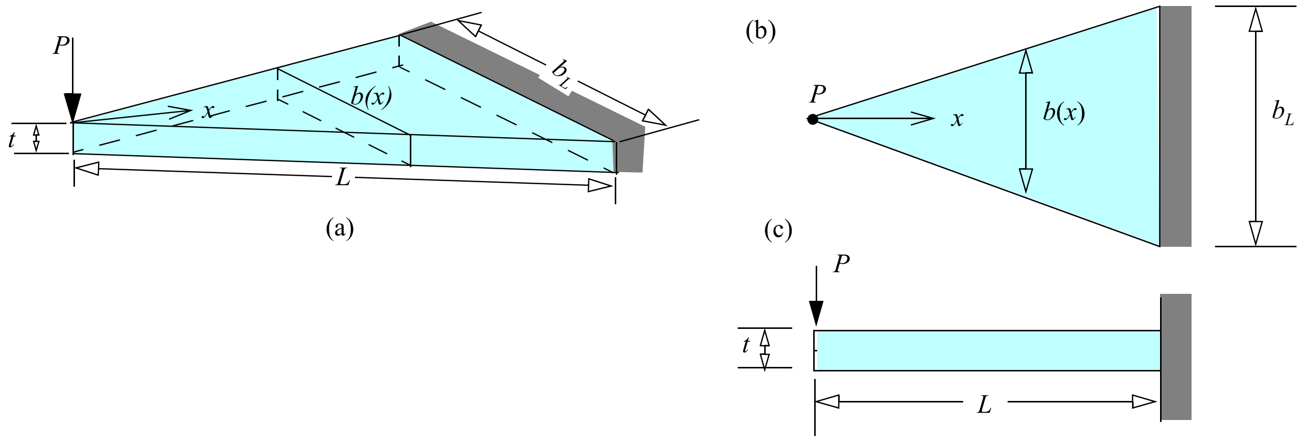


Figure 7.11 (a) Geometry of variable-width beam in Example 7.4. (b) Top view. (c) Front view.

PLAN

The area moment of inertia and the bending moment can also be found of x and substituted into Equation (7.1) to obtain the differential equation. The zero deflection and slope at $x = L$ are the boundary conditions necessary to solve the boundary-value problem for the elastic curve. The maximum deflection will be at $x = 0$ and can be found from the equation of the elastic curve.

SOLUTION

Noting that $b(x)$ is a linear function of x that passes through the origin and has a value of b_L at $x = L$, we obtain $b(x) = b_L x/L$ and the area moment of inertia as

$$I_{zz} = \frac{b(x)t^3}{12} = \left(\frac{b_L t^3}{12L}\right)x \tag{E1}$$

Figure 7.12 shows the free body diagram after an imaginary cut can be made at some location x. By equilibrium of moment at about O, we obtain the internal moment,

$$\boldsymbol{M}_z = -Px \tag{E2}$$

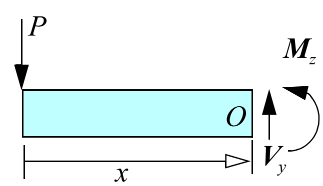


Figure 7.12 Free-body diagram in Example 7.4.

The boundary-value problem can be written as follows:

- **Differential equation:**

$$\frac{d^2 \mathrm{v}}{dx^2} = \frac{M_z}{EI_{zz}} = \frac{-Px}{E(b_L t^3/12L)x} = -\frac{12PL}{Eb_L t^3} \tag{E3}$$

- **Boundary conditions:**

$$v(L) = 0 \tag{E4}$$

$$\frac{dv}{dx}(L) = 0 \tag{E5}$$

Integrating Equation (E3) we obtain

$$\frac{dv}{dx} = -\frac{12PL}{Eb_L t^3}x + c_1 \tag{E6}$$

Substituting Equation (E6) into Equation (E5), we obtain

$$0 = -\frac{12PL}{Eb_L t^3}L + c_1 \qquad \text{or} \qquad c_1 = \frac{12PL^2}{Eb_L t^3} \tag{E7}$$

Substituting Equation (E7) into Equation (E6) and integrating, we obtain

$$v = -\frac{12PL}{Eb_L t^3}\left(\frac{x^2}{2}\right) + \frac{12PL^2}{Eb_L t^3}x + c_2 \tag{E8}$$

Substituting Equation (E8) into Equation (E4), we obtain

$$0 = -\frac{12PL}{Eb_L t^3}\left(\frac{L^2}{2}\right) + \frac{12PL^2}{Eb_L t^3}L + c_2 \qquad \text{or} \qquad c_2 = -\frac{6PL^3}{Eb_L t^3} \tag{E9}$$

The maximum deflection will occur at the free end. Substituting $x = 0$ into Equation (E8) and using Equation (E9) we obtain the maximum deflection.

$$\textbf{ANS.} \qquad v_{max} = -\frac{6PL^3}{Eb_L t^3} \tag{E10}$$

COMMENTS

1. The beam taper must be gradual given the limitation on the theory described in Section 6.2.
2. We can calculate the maximum bending normal stress in any section by substituting $y = t/2$ and Equations (E1) and (E2) into Equation (6.12), to obtain

$$\sigma_{max} = \left| -Px\frac{t/2}{(b_L t^3/12L)x} \right| = \frac{6PL}{b_L t^2} \tag{E11}$$

3. Equation (E11) shows that the maximum bending normal stress is a constant throughout the beam. Such *constant-strength beams* are used in many designs where reduction in weight is a serious consideration. One such design is elaborated in comment 3.
4. In a *leaf spring* (see page 334), each leaf is considered an independent beam that bends about its own neutral axis because there is no restriction to sliding (see Problem 6.20). The variable-width beam is designed for constant strength, and b_L is found using Equation (E11). The width b_L is then divided into n parts, as shown in Figure 7.13*a*. Except for the main leaf A, all other leaf dimensions are found by taking the one-half leaf width on either side of the main leaf. In the assembled spring, the distance in each leaf from the applied load P is the same as in the original variable-width beam shown in Figure 7.11. Hence each leaf has the same allowable strength at all points. If $\bar{b}$ is the width of each leaf and $\bar{L}$ is the total length of the spring, so that $L = \bar{L}/2$, Equations (E10) and (E11) can be rewritten as

$$\delta = \frac{3P\bar{L}^3}{4nE\bar{b}t^3} \qquad \sigma_{max} = \frac{3P\bar{L}}{n\bar{b}t^2} \tag{7.3}$$

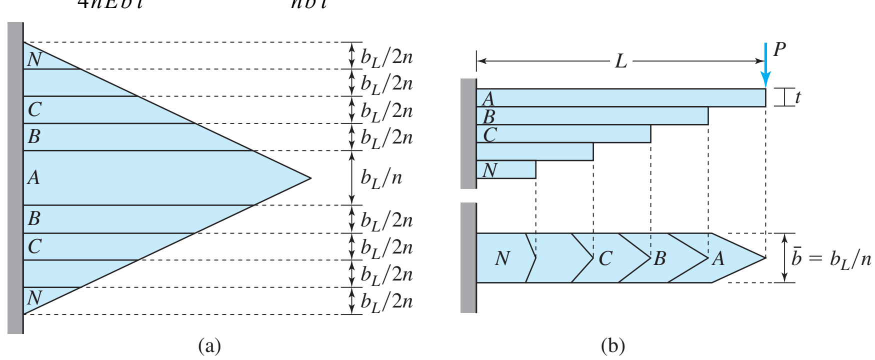


Figure 7.13 Explanation of leaf spring design.

5. The results can be used in design as in Example 3.6.

MoM In Action: Leaf Springs

When it comes to leaf springs, necessity was the mother of invention. Humans realized very early the mechanical advantage of a spring force from bending. For example, most early civilizations had longbows. However, thongs and ropes lose much of their elasticity in dampness and rain. Metals do not, and in 200 B.C.E. Philo of Byzantium proposed using bronze springiness as a source of power. By the early sixteenth century spring-powered clocks attained an accuracy of one minute a day—far better than the weight-driven clocks seen in the towers of Renaissance Italy. The discovery revolutionized navigation, enabling world exploration and European colonial power.

Around the same time, overland travel drove a different kind of spring development. Wagons and carriages felt every bump in the road, and the solution was the first suspension system: leather straps attached to four posts of a chassis suspended the carriage body and isolated it from the chassis. For all its advantages, however, the system did not prevent forward and backward sway, and the high center of gravity left the carriage susceptible to rollover. The problems were significantly reduced by the introduction of *cart springs* (Figure 7.14*a*) or what we call *leaf springs* today. Edouard Phillips (1821–1889) developed the theory of leaf springs (see Example 7.4) while studying the spring suspension in freight trains. It was one of the first applications of the mechanics of materials to engineering design problems.

(a)

(b)

Figure 7.14 Leaf springs in (a) cart; (b) conventional vehicles.

We still use the term *suspension system* today, although cars, trucks, and railways are all supported on springs rather than suspended. To increase bending rigidity, leaf springs have a curve (Figure 7.14*b*). When the curve is elliptical, the springs are referred to as *semi-elliptical* springs. The Ford Model T had a non-elliptical curve, but with the Corvette leaf spring design reached its zenith. Unlike the traditional longitudinal mounting of one spring per wheel, the spring in a Corvette was mounted transversely. This eliminated one leaf spring and significantly reduced the tendency to rollover. A double wishbone design allowed for independent articulation of each wheel. The one-piece fiberglass material practically eliminated fatigue failure, reducing the weight by two thirds compared to steel springs.

With the growth of front wheel drive in the 1970s, automobiles turned instead to coil springs, which require less space and provide each wheel with independent suspension. However, leaf springs continue to be used in trucks and railways, to distribute their heavy loads over larger spans.

Both coil and leaf spring systems are part of a *passive suspension* system, which involve a trade-off between comfort, control, handling, and safety. Those factors are driving newer design systems called *active suspension*, in which the amount of spring force is externally controlled. It took 400 years for leaf springs to reach their zenith, but need has no zenith, and necessity is still the mother of invention in suspension design.

PROBLEM SET 7.1

Second-order boundary-value problems

7.1 For the beam shown in Figure P7.1, determine in terms of *P, L, E,* and *I* (a) the equation of the elastic curve; (b) the deflection of the beam at point *A*.

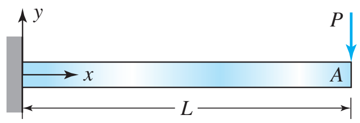


Figure P7.1

7.2 For the beam shown in Figure P7.2, determine in terms of *w, L, E,* and *I* (a) the equation of the elastic curve; (b) the deflection of the beam at point *A*.

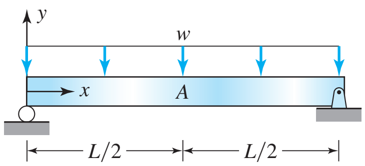


Figure P7.2

7.3 For the beam shown in Figure P7.3, determine in terms of *P, L, E,* and *I* (a) the equation of the elastic curve; (b) the deflection of the beam at point *A*.

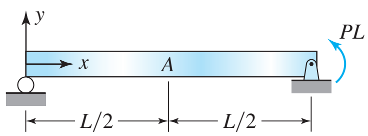


Figure P7.3

7.4 For the beam shown in Figure P7.4, determine in terms of *w, L, E,* and *I* (a) the equation of the elastic curve; (b) the deflection of the beam at point *A*.

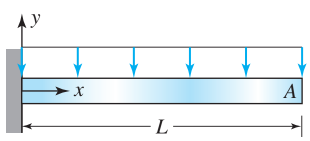


Figure P7.4

7.5 For the beam shown in Figure P7.5, determine in terms of *w, L, E,* and *I* (a) the equation of the elastic curve; (b) the deflection of the beam at point *A*.

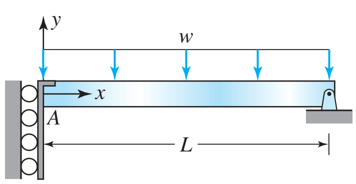


Figure P7.5

7.6 For the beam shown in Figure P7.6, determine in terms of *P, L, E,* and *I* (a) the equation of the elastic curve; (b) the deflection of the beam at point *A*.

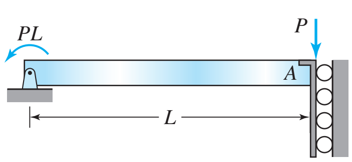


Figure P7.6

7.7 The cantilever beam in Figure P7.7 is acted upon by a distributed bending moment *m* per unit length. Determine (a) the elastic curve in terms of *m*, *E*, *I*, *L*, and *x*; (b) the deflection at $x = L$.

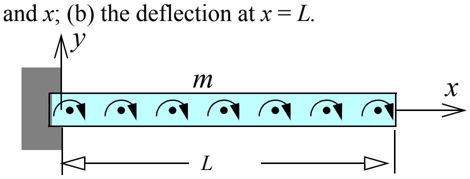


Figure P7.7

7.8 For the beam shown in Figure P7.8 determine the deflection at point *A* in terms of *w, L, E,* and *I.*

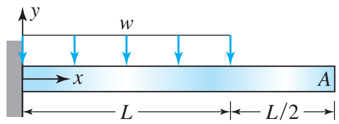


Figure P7.8

7.9 For the beam shown in Figure P7.9 determine the deflection at point *A* in terms of *P, L, E,* and *I.*

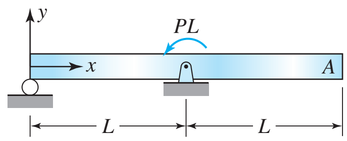


Figure P7.9

7.10 For the beam and loading shown in Figure P7.10, determine the deflection at point *A* in terms of *P, L, E,* and *I.*

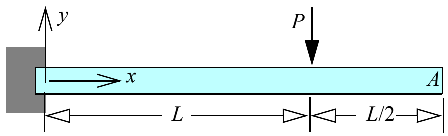


Figure P7.10

7.11 For the beam shown and loading in Figure P7.11, determine the deflection at point *A* in terms of *w, L, E,* and *I.*

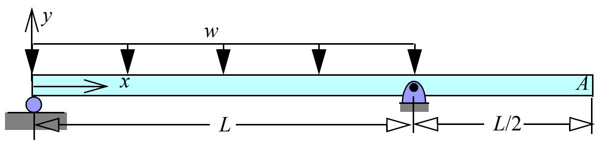


Figure P7.11

7.12 For the beam and loading shown in Figure P7.11, determine the deflection at point *A* in terms of *w, L, E,* and *I*

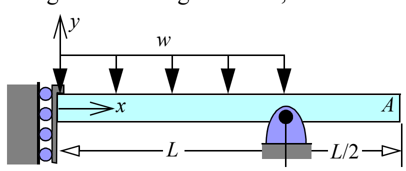


Figure P7.12

7.13 In Table P7.13, v_1 and v_2 represents the deflection in segment *AB* and *BC*. For the beam shown in Figure P7.2, identify all the conditions from Table P7.13 needed to solve for the deflection v(x) at any point on the beam.

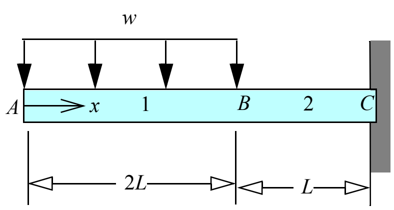


Figure P7.13

TABLE P7.13 **Potential Boundary and Continuity Conditions**

(a) $v_1(0) = 0$	(e) $v_2(2L) = 0$	(i) $v_1(L) = v_2(L)$
(b) $v_1(L) = 0$	(f) $v_2(3L) = 0$	(j) $v_1(2L) = v_2(2L)$
(c) $v_2(L) = 0$	(g) $\frac{dv_1}{dx}(0) = 0$	(k) $\frac{dv_1}{dx}(L) = \frac{dv_2}{dx}(L)$
(d) $v_1(2L) = 0$	(h) $\frac{dv_2}{dx}(3L) = 0$	(l) $\frac{dv_1}{dx}(2L) = \frac{dv_2}{dx}(2L)$

7.14 In Table P7.13, v_1 and v_2 represents the deflection in segment AB and BC. For the beam shown in Figure P7.14, identify all the conditions from Table P7.13 needed to solve for the deflection v(x) at any point on the beam.

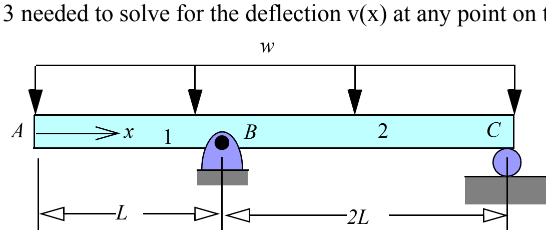


Figure P7.14

7.15 In Table P7.13, v_1 and v_2 represents the deflection in segment *AB* and *BC*. For the beam shown in Figure P7.15, identify all the conditions from Table P7.13 needed to solve for the deflection v(x) at any point on the beam.

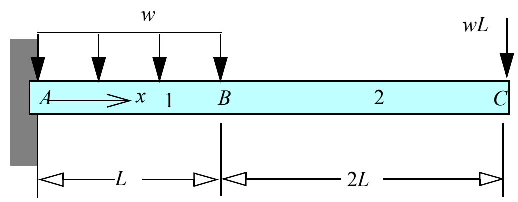


Figure P7.15

7.16 In Table P7.13, v_1 and v_2 represents the deflection in segment *AB* and *BC*. For the beam shown in Figure P7.16, identify all the conditions from Table P7.13 needed to solve for the deflection v(x) at any point on the beam.

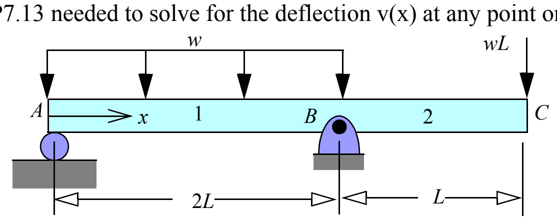


Figure P7.16

7.17 For the beam and loading shown in Figure P7.17, determine in terms of *w, L, E,* and *I* (a) the equation of the elastic curve; (b) the deflection at $x = L$.

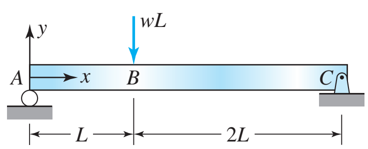


Figure P7.17

7.18 For the beam and loading shown in Figure P7.18, determine in terms of *w, L, E,* and *I* (a) the equation of the elastic curve; (b) the deflection at $x = L$.

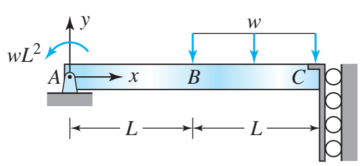


Figure P7.18

7.19 For the beam and loading shown in Figure P7.19 determine in terms of *w, L, E,* and *I* (a) the equation of the elastic curve; (b) the deflection at $x = L$.

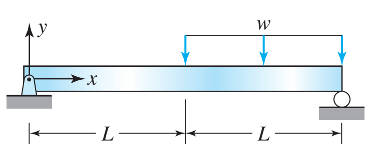


Figure P7.19

7.20 For the beam and loading shown in Figure P7.20 determine in terms of *w, L, E,* and *I* (a) the equation of the elastic curve; (b) the deflection at $x = L$.

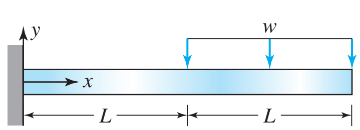


Figure P7.20

7.21 A simply supported beam in Figure P7.21 is acted upon by a distributed bending moment *m* per unit length. Determine (a) the elastic curve in terms of *m*, *E*, *I*, *L*, and *x*; (b) deflection at $x = L$.

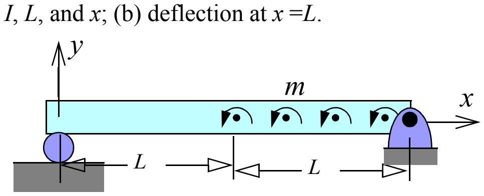


Figure P7.21

7.22 A diver weighing 200 lb stands at the edge of the diving board as shown in Figure 7.22. The diving board cross section is 16 in.x 1 in. and has a modulus of elasticity of 1500 ksi. Determine the maximum deflection in the diving board.

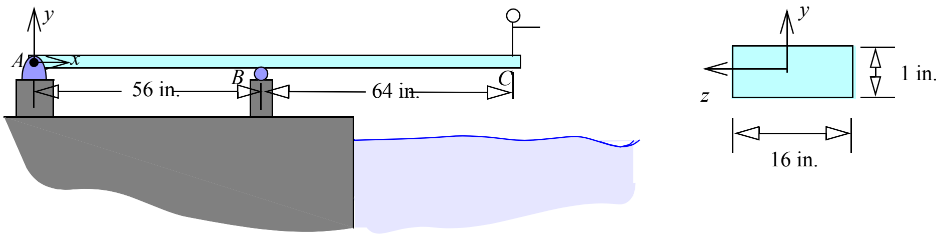


Figure P7.22

7.23 For the beam and loading shown in Figure P7.23, write the boundary-value problem for determining the deflection of the beam at any point *x*. Assume *EI* is constant. Do not integrate or solve.

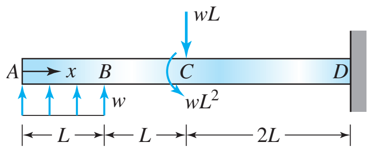


Figure P7.23

7.24 For the beam shown in Figure P7.24, write the boundary-value problem for determining the deflection of the beam at any point *x*. Assume *EI* is constant. Do not integrate or solve.

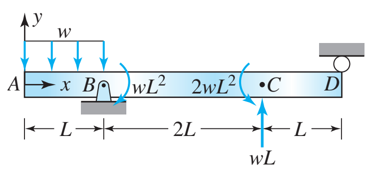


Figure P7.24

Variable area moment of inertia

7.25 A cantilever beam with variable depth $h(x)$ and constant width b is shown in Figure P7.25. The beam is to have a constant strength σ. In terms of b, L, E, x, and σ, determine (a) the variation of $h(x)$; (b) the maximum deflection.

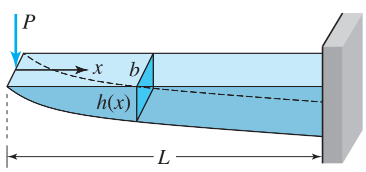


Figure P7.25

7.26 A cantilever tapered circular beam with variable radius $R(x)$ is shown in Figure P7.26.The beam is to have a constant strength σ. In terms of L, E, x, and σ, determine (a) the variation of $R(x)$; (b) the maximum deflection.

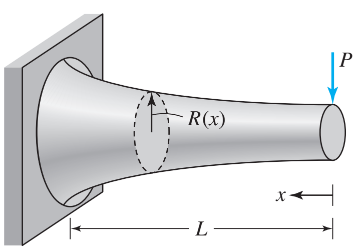


Figure P7.26

7.27 For the tapered beam shown in Figure P7.27, determine the maximum bending normal stress and the maximum deflection in terms of E, w, b, h_0, and L.

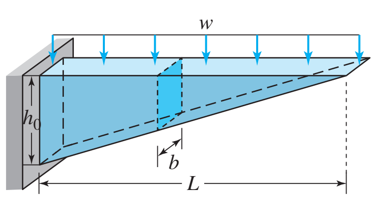


Figure P7.27

7.28 For the tapered circular beam shown in Figure P7.28, determine the maximum bending normal stress and the maximum deflection in terms of E, P, d_0, and L.

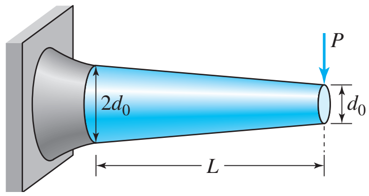


Figure P7.28

7.29 The 2-in. × 8-in. wooden beam of rectangular cross section shown in Figure P7.29 is braced at the support using 2-in. × 1-in. wooden pieces. The modulus of elasticity of wood is 2000 ksi. Determine the maximum bending normal stress and the maximum deflection.

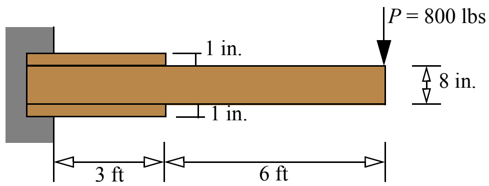


Figure P7.29

7.30 A 2 in. x 8 in. wooden rectangular cross-section beam is braced the near the load point using 2 in. x 1 in. wooden pieces as shown in Figure P7.30. The load is applied at the mid point of the beam. The modulus of elasticity of wood is 2,000 ksi. Determine the maximum bending normal stress and the maximum deflection. (Hint: Use symmetry about mid point to reduce calculations)

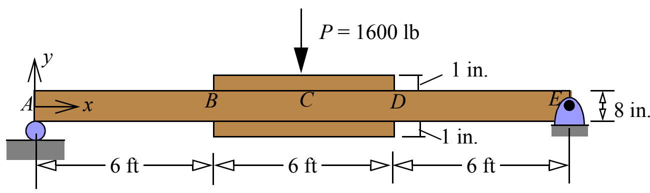


Figure P7.30

Stretch Yourself

7.31 To reduce weight of a metal beam the flanges are made of steel $E = 200$ GPa and the web of aluminum $E = 70$ GPa as shown in Figure P7.31. Determine the maximum deflection of the beam.

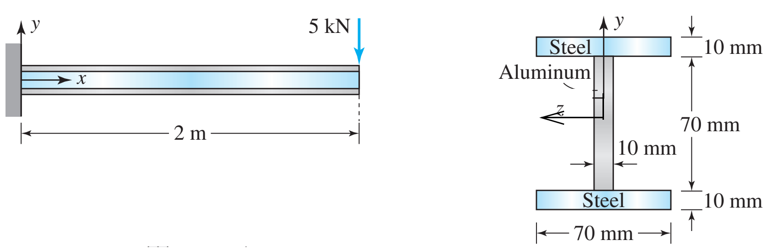


Figure P7.31

7.2 FOURTH-ORDER BOUNDARY-VALUE PROBLEM

We were able to solve for the deflection of a beam in Section 7.1 using second-order differential equations because we could find $\boldsymbol{M}_z$ as a function of x. In statically indeterminate beams, the internal moment determined from static equilibrium will contain some unknown reactions in the moment expression. Also, if the distributed load p_y is not uniform or linear but a more complicated function, then finding the internal moment $\boldsymbol{M}_z$ as a function of x may be difficult. In either case it may be preferable to start from an alternative equation. We can substitute Equation (7.1) into Equation (6.17), (that is, into $d\boldsymbol{M}_z/dx = -\boldsymbol{V}_y$) and then substitute the result into Equation (E6.18), (that is, $d\boldsymbol{V}_y/dx = -p_y$) to obtain

$$\boldsymbol{V}_y = -\frac{d}{dx}\left(EI_{zz}\frac{d^2v}{dx^2}\right) \tag{7.4}$$

$$\frac{d^2}{dx^2}\left(EI_{zz}\frac{d^2v}{dx^2}\right) = p_y \tag{7.5}$$

If the bending rigidity EI_{zz} is constant, then it can be taken outside the differentiation. However, if the beam is tapered, then I_{zz} is a function of x, and the form given in Equations (7.4) and (7.5) must be used.

7.2.3 Boundary Conditions

The deflection v(x) can be obtained by integrating Equation (7.5), but the fourth-order differential equation will generate four integration constants. To determine these constants, four boundary conditions are needed. The integration of Equation (7.5) will yield V_y of Equation (7.4), which on integration would yield M_z of Equation (7.1), which on integration would in turn yield v and dv/dx. Thus boundary conditions could be imposed on any of the four quantities v, dv/dx, M_z, and V_y.

To understand how these conditions are determined, we generalize a principle discussed in statics for determining the reaction force and/or moments. Recall how we determine reaction forces and moments at the supports in drawing free-body diagrams:

- If a point cannot move in a given direction, then a reaction force opposite to the direction acts at that support point.
- If a line cannot rotate about an axis in a given direction, then a reaction moment opposite to the direction acts at that support.

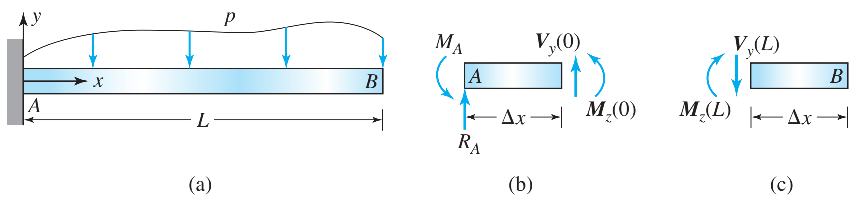


Figure 7.15 Example demonstrating grouping of boundary conditions.

Consider the cantilever beam with an arbitrarily varying distributed load shown in Figure 7.15*a*. We make an imaginary cut very close to the support at A (at an infinitesimal distance Δx) and draw the free-body diagram as shown in Figure 7.15*b*. The internal shear force and the internal moment are drawn according to our sign convention. Notice that the distributed force is not shown because as Δx goes to zero, the contribution of the distributed force will drop out from the equilibrium equations. By equilibrium we obtain $V_y(0) = -R_A$ and $M_z(0) = -M_A$. Thus if a point cannot move—that is, the deflection v is zero at a point—then the shear force is not known, because the reaction force is not known. Similarly if a line cannot rotate around an axis passing through a point, $dv/dx = 0$, and the internal moment is not known because the reaction moment is not known.

The reverse is equally true. Consider the free-body diagram constructed after making an imaginary cut at an infinitesimal distance from end B, as shown Figure 7.15*c*. By equilibrium we obtain $V_y(L) = 0$ and $M_z(L) = 0$. However, the free end can deflect and rotate by any amount dictated by the loading. Thus, when we specify a value of shear force, then we cannot specify displacement. And when we specify a value of internal moment at a point, then we cannot specify rotation. We can thus place the quantities v, dv/dx, M_z, and V_y:

- *Group 1:* At a boundary point either the deflection v can be specified or the internal shear force V_y can be specified, but not both.
- *Group 2:* At a boundary point either the slope dv/dx can be specified or the internal bending moment M_z can be specified, but not both.

Two conditions are specified at each end of the beam, generating four boundary conditions. One condition is chosen from each group. Stated succinctly, the boundary conditions at each end of the beam are

$$\begin{array}{lccc} \bullet\ \textit{Group 1:} & v & \text{or} & V_y \\ & & \text{and} & \\ \bullet\ \textit{Group 2:} & \dfrac{dv}{dx} & \text{or} & M_z \end{array} \tag{7.6}$$

From Figure 7.15*c* we concluded that the shear force and the bending moment at the free end were zero. This conclusion can be reached by inspection without drawing a free-body diagram. If at the end there were a point force or a point moment, then clearly the magnitude of the shear force would equal the point force, and the magnitude of the internal moment would equal the point moment. Again, we can reach this conclusion without drawing a free-body diagram. But to get the correct sign of V_y and M_z we need a free-body diagram, with the internal quantities drawn according to our sign convention. We address the issue in Section 7.2.5.

7.2.4 Continuity and Jump Conditions

Suppose there is a point force or a point moment at x_j, or that the distributed force is given by different functions on the left and right of x_j. Then, again, the displacement will be represented by different functions on the left and right of x_j.

Thus we have two fourth-order differential equations, and their integration constants will require eight conditions:

- Four conditions are the boundary conditions discussed in Section 7.2.3.
- Two additional conditions are the continuity conditions at x_j discussed in Section 7.1.2.
- The remaining two conditions are the equilibrium equations on V_y and M_z at x_j.

The equilibrium conditions on V_y and M_z at x_j are jump conditions due to a point force or a point moment to be discussed in the next section.

7.2.5 Use of Template in Boundary Conditions or Jump Conditions

We discussed the concept of templates in drawing shear–moment diagrams in Section 6.4.2. Here we discuss it in determining the boundary conditions on V_y and M_z and jumps in these internal quantities due to a point force or a point moment.

Recall that a template is a small segment of a beam on which a point moment M_j and a point force F_j are drawn (Δx tends to zero in Figure 7.16). F_j and M_j could be applied or reactive forces and moments and their directions are arbitrary. The ends at $+\Delta x$ and $-\Delta x$ represent the imaginary cut just to the left and just to the right of the point forces and point moments. The internal shear force and the internal bending moment on these imaginary cuts are drawn according to our sign convention, as discussed in Section 6.2.6. Writing the equilibrium equations for this $2\Delta x$ segment of the beam, we obtain the template equations,

$$V_2(x_j) - V_1(x_j) = -F_j \tag{7.7.a}$$

$$M_2(x_j) - M_1(x_j) = M_j \tag{7.7.b}$$

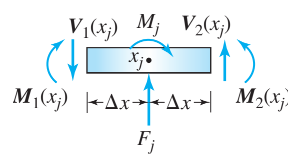


Figure 7.16 Template at x_j.

The moment equation does not contain the moment due to the forces because these moments will go to zero as Δx goes to zero.

If the point force on the beam is in the direction of F_j shown on the template, the template equation for force is used as given. If the point force on the beam is opposite to the direction of F_j shown on the template, then the template equation is used by changing the sign of F_j. The template equation for the moment is used in a similar fashion.

- If x_j is a *left* boundary point, then there is no beam left of x_j. Hence V_1 and M_1 are zero and we obtain the boundary conditions on V_y and M_z from V_2 and M_2.
- If x_j is a *right* boundary point, then there is no beam right of x_j. Hence V_2 and M_2 are zero and we obtain the boundary conditions on V_y and M_z from V_1 and M_1.
- If x_j is in between the ends of the beam, then the jump in shear force and internal moment is calculated using the template equations.

An alternative to the templates is to draw free-body diagrams after making imaginary cuts at an infinitesimal distance from the point force and writing equilibrium equations. Example 7.5 demonstrates the use of the template. Examples 7.6 and 7.7 demonstrate the use of free-body diagrams to determine the boundary conditions or the jump in internal quantities.

EXAMPLE 7.5

The bending rigidity of the beam shown in Figure 7.17 is 135 (10^6) lbs·in.^{2}, and the displacements of the beam in segments *AB* (v_1) and *BC* (v_2) are as given below. Determine (a) the reactions at the left wall at *A*; (b) the reaction force at *B* and the applied moment M_B.

$$v_1 = 5(x^3 - 20x^2)10^{-6} \text{ in.} \qquad 0 \le x \le 20 \text{ in.}$$

$$v_2 = 10(x^3 - 30x^2 + 200x)10^{-6} \text{ in.} \qquad 20 \text{ in.} \le x \le 40 \text{ in.}$$

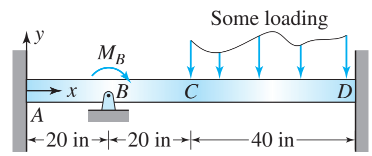


Figure 7.17 Beam in Example 7.5.

PLAN

By differentiating the given displacement functions and using Equations (7.1) and (7.4), we can find the bending moment M_z and the shear force V_y in segments *AB* and *BC*. (a) Using the template in Figure 7.16, we can find the reactions at *A* from the values of V_y and M_z at $x = 0$. (b) Using the template in Figure 7.16, we can find the reaction force and the applied moment at *B* from the values of V_y and M_z before and after $x = 20$ in.

SOLUTION

The shear force calculation requires the third derivative of the displacement functions. The functions v_1 and v_2 can be differentiated three times:

$$\frac{dv_1}{dx} = 5(3x^2 - 40x)(10^{-6}) \tag{E1}$$

$$\frac{d^2v_1}{dx^2} = 5(6x - 40)(10^{-6}) \text{ in.}^{-1} \tag{E2}$$

$$\frac{d^3v_1}{dx^3} = (5)(6)(10^{-6}) = 30(10^{-6}) \text{ in.}^{-2} \tag{E3}$$

$$\frac{dv_2}{dx} = 10(3x^2 - 60x + 200)(10^{-6}) \tag{E4}$$

$$\frac{d^2v_2}{dx^2} = 10(6x - 60)(10^{-6}) \text{ in.}^{-1} \tag{E5}$$

$$\frac{d^3v_2}{dx^3} = (10)(6)(10^{-6}) = 60(10^{-6}) \text{ in.}^{-2} \tag{E6}$$

From Equations (7.1), (E2), and (E5), the internal moment is

$$M_{z_1} = EI_{zz}\frac{d^2v_1}{dx^2} = [135(10^6) \text{ lbs.in.}^2][5(6x - 40)(10^{-6}) \text{ in.}^{-1}] = 675(6x - 40) \text{ in.·lbs} \tag{E7}$$

$$M_{z_2} = EI_{zz}\frac{d^2v_2}{dx^2} = [135(10^6) \text{ lbs.in.}^2][10(6x - 60)(10^{-6}) \text{ in.}^{-1}] = 1350(6x - 60) \text{ in.·lbs} \tag{E8}$$

From Equations (7.4), (E3), and (E6), the shear force is

$$V_{y_1} = EI_{zz}\frac{d^3v_1}{dx^3} = [135(10^6) \text{ lbs.in.}^2][30(10^{-6}) \text{ in.}^{-2}] = 4050 \text{ lbs} \tag{E9}$$

$$V_{y_2} = EI_{zz}\frac{d^3v_2}{dx^3} = [135(10^6) \text{ lbs.in.}^2][60(10^{-6}) \text{ in.}^{-2}] = 8100 \text{ lbs} \tag{E10}$$

The internal moment and shear force at *A* can be found by substituting $x = 0$ into Equations (E7) and (E9),

$$M_{z_1}(0) = 675(-40) = -27{,}000 \text{ in.·lbs} \qquad V_{y_1}(0) = 4050 \; lbs \tag{E11}$$

The internal moment and shear force just before and after *B* can be found by substituting $x = 20$ into Equations (E7) through (E10),

$$M_{z_1}(20) = 675[(6)(20) - 40] = 54{,}000 \text{ in.}\cdot\text{lbs} \qquad M_{z_2}(20) = 1350[(6)(20) - 60] = 81{,}000 \text{ in.}\cdot\text{lbs} \tag{E12}$$

$$V_{y_1}(20) = 4050 \text{ lbs} \qquad V_{y_2}(20) = 8100 \text{ lbs} \tag{E13}$$

Figure 7.18 shows the free-body diagram of the entire beam. It also shows the template of Figure 7.16 for convenience.

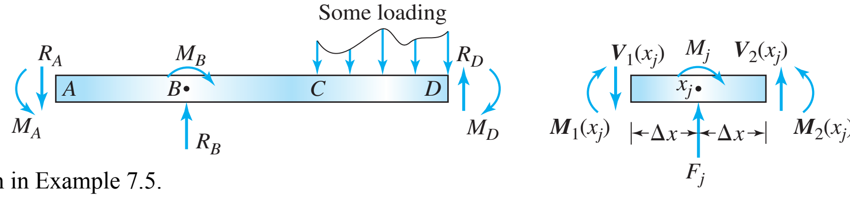


Figure 7.18 Free-body diagram of entire beam in Example 7.5.

If we compare the reaction force at A to F_j and the reaction moment to M_j in Figure 7.16, we obtain $F_j = -R_A$ and $M_j = -M_A$. As point A is the left end of the beam, $V_1(x_j)$ and $M_1(x_j)$ are zero on the template, and $V_2(x_j) = V_{y_1}(0)$ and $M_2(x_j) = M_{z_1}(0)$. From the template equation we obtain $R_A = V_{y_1}(0)$ and $M_A = -M_{z_1}(0)$. Substituting Equation (E11), we obtain

ANS. $R_A = 4050$ lbs $\qquad M_A = 27{,}000$ in. · lbs

If we compare the reaction force at B to F_j and the applied moment to M_j in Figure 7.16, we obtain $F_j = R_B$ and $M_j = M_B$. Substituting for $x_j = 20$ in. and using Equations (E12) through (E13), we obtain R_B and M_B,

$$R_B = V_1(20) - V_{y_2}(20) = 4050 \text{ lbs} - 8100 \text{ lbs} = -4500 \text{ lbs} \tag{E14}$$

$$M_B = M_{z_2}(20) - M_{z_1}(20) = 81{,}000 \text{ in.}\cdot\text{lbs} - 54{,}000 \text{ in.}\cdot\text{lbs} = 27{,}000 \text{ in.}\cdot\text{lbs} \tag{E15}$$

ANS. $R_B = -4500$ lbs $\qquad M_B = 27{,}000$ in.·lbs

COMMENTS

1. An alternative to the use of the template is to draw a free-body diagram after making imaginary cuts at an infinitesimal distance from the point forces, as shown in Figure 7.19. The internal forces and moments must be drawn according to our sign convention. By writing equilibrium equations the required quantities can be found.

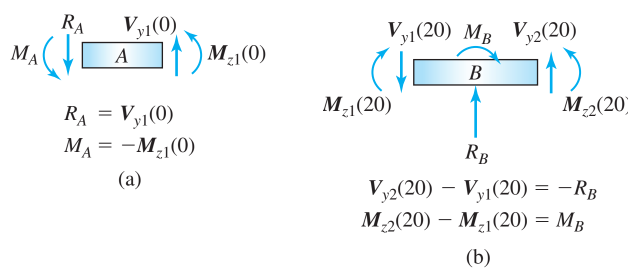


Figure 7.19 Alternative to template.

2. The free-body diagram of the entire beam in Figure 7.18 is not necessary. From the template equations, the force F_j and the moment M_j with the correct signs can be found. If F_j and M_j are positive, then R_B and M_B will be in the direction shown on the template. If these quantities are negative, then the direction is opposite.
3. This problem demonstrates how we (i) determine the conditions on shear force and bending moment and (ii) relating these internal quantities to the reaction forces and moments. The same basic principles apply when the displacement functions have to be determined first, as we see next.

EXAMPLE 7.6

In terms of E, I, w, L, and x, determine (a) the elastic curve; (b) the reaction force at A in Figure 7.20.

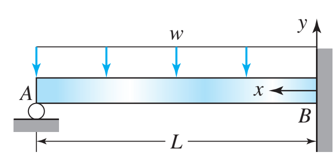


Figure 7.20 Beam and loading in Example 7.6.

METHOD 1 PLAN: FOURTH-ORDER DIFFERENTIAL EQUATION

(a) Noting that the distributed force is in the negative y direction, we can substitute $p_y = -w$ in Equation (7.5) and write the fourth-order differential equation. The two boundary conditions at A are zero deflection and zero moment, and the two boundary conditions at B are zero deflection and zero slope. We can solve the boundary-value problem and obtain the elastic curve. (b) We can draw a free-body diagram after making an imaginary cut just to the right of A and relate the reaction force to the shear force. We can find the shear force at point A by substituting $x = L$ in the solution obtained in part (a).

SOLUTION

(a) The boundary-value problem statement can be written below following the description in the Plan.

- **Differential equation:**

$$\frac{d^2}{dx^2}\left(EI_{zz}\frac{d^2v}{dx^2}\right) = -w \tag{E1}$$

- **Boundary conditions:**

$$v(0) = 0 \tag{E2}$$

$$\frac{dv}{dx}(0) = 0 \tag{E3}$$

$$v(L) = 0 \tag{E4}$$

$$EI_{zz}\frac{d^2v}{dx^2}(L) = 0 \tag{E5}$$

Integrating Equation (E1) twice,

$$\frac{d}{dx}\left(EI_{zz}\frac{d^2v}{dx^2}\right) = -wx + c_1 \tag{E6}$$

$$EI_{zz}\frac{d^2v}{dx^2} = -\frac{wx^2}{2} + c_1x + c_2 \tag{E7}$$

Substituting Equation (E7) into Equation (E5), we obtain

$$c_1L + c_2 = \frac{wL^2}{2} \tag{E8}$$

Integrating Equation (E7), we obtain

$$EI_{zz}\frac{dv}{dx} = -\frac{wx^3}{6} + c_1\frac{x^2}{2} + c_2x + c_3 \tag{E9}$$

Substituting Equation (E9) into Equation (E3), we obtain

$$c_3 = 0 \tag{E10}$$

Substituting Equation (E10) and integrating Equation (E9), we obtain

$$EI_{zz}v = -\frac{wx^4}{24} + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_4 \tag{E11}$$

Substituting Equation (E11) into Equation (E2), we obtain

$$c_4 = 0 \tag{E12}$$

Substituting Equations (E12) and (E11) into Equation (E4), we obtain

$$\frac{c_1L^3}{6} + \frac{c_2L^2}{2} = \frac{wL^4}{24} \tag{E13}$$

Solving Equations (E8) and (E13) simultaneously, we obtain

$$c_1 = \frac{5wL}{8} \qquad c_2 = -\frac{wL^2}{8} \tag{E14}$$

Substituting Equations (E12) and (E14) into Equation (E11) and simplifying, we obtain the elastic curve,

$$\text{ANS.} \qquad v(x) = -\frac{w}{48EI_{zz}}(2x^4 - 5Lx^3 + 3L^2x^2) \tag{E15}$$

Dimension check: Note that all terms in parentheses on the right-hand side of Equation (E15) have the dimension of length to the power of 4, or $O(L^4)$. Thus Equation (E15) is dimensionally homogeneous. But we can also check whether the left-hand side and any one term of the right-hand side have the same dimension:

$$w \to O\left(\frac{F}{L}\right) \qquad x \to O(L) \qquad E \to O\left(\frac{F}{L^2}\right) \qquad I_{zz} \to O(L^4) \qquad v \to O(L) \qquad \frac{wx^4}{EI_{zz}} \to O\left(\frac{(F/L)L^4}{(F/L^2)O(L^4)}\right) \to O(L) \to \text{checks}$$

(b) We make an imaginary cut just to the right of point A (at an infinitesimal distance) and draw the free-body diagram of the left part using the sign convention in Section 6.2.6, as shown in Figure 7.21. By force equilibrium in the y direction, we can relate the shear force at A to the reaction force at A,

$$R_A = \boldsymbol{V}_A = \boldsymbol{V}_y(L) \tag{E16}$$

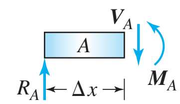


Figure 7.21 Infinitesimal equilibrium element at A in Example 7.6.

From Equations (7.4), (E6), and (E14), the shear force is

$$\boldsymbol{V}_y(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\left(EI_{zz}\frac{\mathrm{d}^2\mathrm{v}}{\mathrm{d}x^2}\right) = wx - \frac{5wL}{8} \tag{E17}$$

Substituting Equation (E17) into Equation (E16), we obtain the reaction at A.

ANS. $R_A = \dfrac{3wL}{8}$

METHOD 2 PLAN: SECOND-ORDER DIFFERENTIAL EQUATION

We can make an imaginary cut at some arbitrary location x and use the left part to draw the free-body diagram. The moment expression will contain the reaction force at A as an unknown. The second-order differential equation, Equation (7.1), would generate two integration constants, leading to a total of three unknowns. We need three conditions: the displacement at A is zero, and the displacement and slope at B are both zero. Solving the boundary-value problem, we can obtain the elastic curve and the unknown reaction force at A.

SOLUTION

We make an imaginary cut at a distance x from the right wall and take the left part of length $L - x$ to draw the free-body diagram using the sign convention for internal quantities discussed in Section 6.2.6 as shown in Figure 7.22.

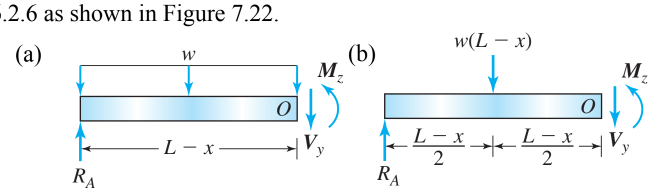


Figure 7.22 Free-body diagram in Example 7.6.

Balancing the moment at point O, we obtain the moment expression,

$$\boldsymbol{M}_z - R_A(L-x) + w\frac{(L-x)^2}{2} = 0 \qquad \text{or} \qquad \boldsymbol{M}_z = R_A(L-x) - \frac{w}{2}(L^2 + x^2 - 2Lx) \tag{E1}$$

Substituting into Equation (7.1) and writing the boundary conditions, we obtain the following boundary-value problem:

- **Differential equation:**

$$EI_{zz}\frac{d^2\mathrm{v}}{dx^2} = R_A(L-x) - \frac{w}{2}(L^2 + x^2 - 2Lx) \tag{E2}$$

- **Boundary conditions:**

$$\mathrm{v}(0) = 0 \tag{E3}$$

$$\frac{d\mathrm{v}}{dx}(0) = 0 \tag{E4}$$

$$\mathrm{v}(L) = 0 \tag{E5}$$

Integrating Equation (E2), we obtain

$$EI_{zz}\frac{\mathrm{dv}}{\mathrm{d}x} = R_A\left(Lx - \frac{x^2}{2}\right) - \frac{w}{2}\left(L^2x + \frac{x^3}{3} - Lx^2\right) + c_1 \tag{E6}$$

Substituting Equation (E6) into Equation (E4), we obtain

$$c_1 = 0 \tag{E7}$$

Substituting Equation (E7) and integrating Equation (E6), we obtain

$$EI_{zz}\mathrm{v} = R_A\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) - \frac{w}{2}\left(\frac{L^2x^2}{2} + \frac{x^4}{12} - \frac{Lx^3}{3}\right) + c_2 \tag{E8}$$

Substituting Equation (E8) into Equation (E3), we obtain

$$c_2 = 0 \tag{E9}$$

Substituting Equations (E8) and (E9) into Equation (E5), we obtain

$$R_A\left(\frac{L^3}{2}-\frac{L^3}{6}\right)-\frac{w}{2}\left(\frac{L^4}{2}+\frac{L^4}{12}-\frac{L^4}{3}\right) = 0 \quad \text{or} \quad \text{ANS.} \quad R_A = \frac{3wL}{8} \tag{E10}$$

Substituting Equations (E9) and (E10) into Equation (E8) and simplifying, we obtain v(x).

$$\textbf{ANS.} \qquad \mathrm{v}(x) = -\frac{w}{48EI_{zz}}(2x^4 - 5Lx^3 + 3L^2x^2) \tag{E11}$$

COMMENTS

1. Method 2 has less algebra than Method 1 and should be used whenever possible.
2. Suppose that in drawing the free-body diagram for calculating the internal moment, we had taken the right-hand part. Then we would have two unknowns rather than one—the wall reaction force and moment in the expression for moment. In such a case we would have to eliminate one of the unknowns using the static equilibrium equation for the entire beam. In other words, in statically indeterminate problems, the internal moment should contain a number of unknown reactions equal to the degree of static redundancy.
3. The moment boundary condition given by in Method 1 is implicitly satisfied. We can confirm this by substituting $x = L$ in Equation (E1).

EXAMPLE 7.7

A light pole is subjected to a wind pressure that varies as a quadratic function, as shown in Figure 7.23. In terms of E, I, w, L, and x, determine (a) the deflection at the top of the pole; (b) the ground reactions.

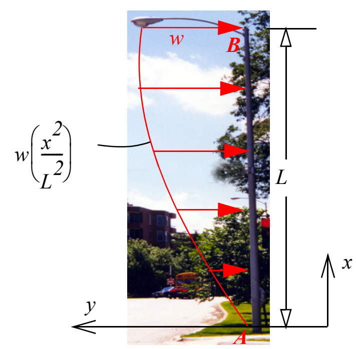


Figure 7.23 Beam and loading in Example 7.7.

PLAN

(a) Finding the moment as a function of x by static equilibrium is difficult for this statically determinate problem. We can use the fourth-order differential equation, Equation (7.5). We have four boundary conditions: the deflection and slope at A are zero, and the moment and shear force at B are zero. We can then solve the boundary-value problem and determine the elastic curve. By substituting $x = L$ in the elastic curve equation, we can obtain the deflection at the top of the pole. (b) By making an imaginary cut just above point A, we can relate the internal shear force and the internal moment at point A to the reactions at A. By substituting $x = 0$ in the moment and shear force expressions, we can obtain the shear force and moment values at point A.

SOLUTION

The boundary-value problem below can be written as described in the Plan.

- **Differential equation:**

$$\frac{d^2}{dx^2}\left(EI_{zz}\frac{d^2\mathrm{v}}{dx^2}\right) = -w\left(\frac{x^2}{L^2}\right) \tag{E1}$$

- **Boundary conditions:**

$$\mathrm{v}(0) = 0 \tag{E2}$$

$$\frac{d\mathrm{v}}{dx}(0) = 0 \tag{E3}$$

$$EI_{zz}\left.\frac{d^2\mathrm{v}}{dx^2}\right|_{x=L} = 0 \tag{E4}$$

$$\left.\frac{d}{dx}\left(EI_{zz}\frac{d^2\mathrm{v}}{dx^2}\right)\right|_{x=L} = 0 \tag{E5}$$

Integrating Equation (E1), we obtain

$$\frac{d}{dx}\left(EI_{zz}\frac{d^2v}{dx^2}\right) = -\frac{wx^3}{3L^2} + c_1 \tag{E6}$$

Substituting Equation (E6) into Equation (E5), we obtain

$$c_1 = \frac{wL}{3} \tag{E7}$$

Substituting Equation (E7) into Equation (E6) and integrating, we obtain

$$EI_{zz}\frac{d^2v}{dx^2} = -\frac{wx^4}{12L^2} + \frac{wL}{3}x + c_2 \tag{E8}$$

Substituting Equation (E8) into Equation (E4), we obtain

$$c_2 = -\frac{wL^2}{4} \tag{E9}$$

Substituting Equation (E9) into Equation (E8) and integrating, we obtain

$$EI_{zz}\frac{dv}{dx} = -\frac{wx^5}{60L^2} + \frac{wLx^2}{6} - \frac{wL^2x}{4} + c_3 \tag{E10}$$

Substituting Equation (E10) into Equation (E3), we obtain

$$c_3 = 0 \tag{E11}$$

Substituting Equation (E11) into Equation (E10) and integrating, we obtain

$$EI_{zz}\frac{dv}{dx} = -\frac{wx^6}{360L^2} + \frac{wLx^3}{18} - \frac{wL^2x^2}{8} + c_4 \tag{E12}$$

Substituting Equation (E12) into Equation (E2), we obtain

$$c_4 = 0 \tag{E13}$$

Substituting Equation (E13) into Equation (E12) and simplifying, we obtain

$$v(x) = -\frac{w}{360EI_{zz}L^2}(x^6 - 20L^3x^3 + 45L^4x^2) \tag{E14}$$

Dimension check: Note that all terms in parentheses on the right-hand side of Equation (E14) have the dimension of length to the power of 6, or, $O(L^6)$. Thus Equation (E14) is dimensionally homogeneous. But we can also check whether the left-hand side and any one term of the right-hand side have the same dimension:

$$w \rightarrow O\left(\frac{F}{L}\right) \qquad x \rightarrow O(L) \qquad E \rightarrow O\left(\frac{F}{L^2}\right) \qquad I_{zz} \rightarrow O(L^4) \qquad v \rightarrow O(L) \qquad \frac{wx^6}{EI_{zz}L^2} \rightarrow O\left(\frac{(F/L)L^6}{(F/L^2)L^4L^2}\right) \rightarrow O(L) \rightarrow \text{checks}$$

(a) Substituting $x = L$ into , we obtain the deflection at the top of the pole.

ANS. $$v(L) = -\frac{13wL^4}{180EI_{zz}}$$

(b) We make an imaginary cut just above point A $(\Delta x \rightarrow 0)$ and take the bottom part to draw the free-body diagram shown in Figure 7.24. By equilibrium of forces and moments, we can relate the reaction force R_A and the reaction moment M_A to the internal shear force and the internal bending moment at point A,

$$\boldsymbol{M}_z(0) = -M_A \qquad \boldsymbol{V}_y(0) = -R_A \tag{E15}$$

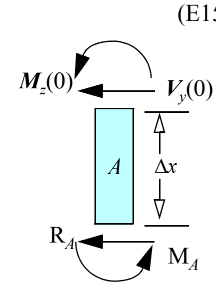


Figure 7.24 Free body diagram of an infinitesimal element at A in Example 7.7.

Substituting Equations (E7) and (E6) into Equation (7.4) and Equations (E9) and (E8) into Equation (7.1), we can obtain the shear force and bending moment expressions,

$$\boldsymbol{M}_z(x) = -\frac{wx^4}{12L^2} + \frac{wL}{3}x - \frac{wL^2}{4} \tag{E16}$$

$$V_y(x) = \frac{wx^3}{3L^2} - \frac{wL}{3} \tag{E17}$$

Substituting Equations (E16) and (E17) into Equation (E15), we obtain the reaction force and the reaction moment.

ANS. $R_A = \dfrac{wL}{3}$ $\quad M_A = \dfrac{wL^2}{4}$

COMMENTS

1. The directions of R_A and M_A can be checked by inspection, as these are the directions necessary for equilibrium of the externally distributed force.
2. The free-body diagram in Figure 7.24, the reaction force R_A and the reaction moment M_A can be drawn in any direction, but the internal quantities V_y and M_z must be drawn according to the sign convention in Section 6.2.6. Irrespective of the direction in which R_A and M_A are drawn, the final answer will be as given. The sign in the equilibrium equations, Equation (E15), will account for the assumed directions of the reactions.

PROBLEM SET 7.2

Fourth-order boundary-value problems

7.32 The displacement in the y direction in segment AB, shown in Figure P7.32, was found to be $v(x) = (20x^3 - 40x^2)10^{-6}$ in. If the bending rigidity is 135×10^6 lb·in.², determine the reaction force and the reaction moment at the wall at A.

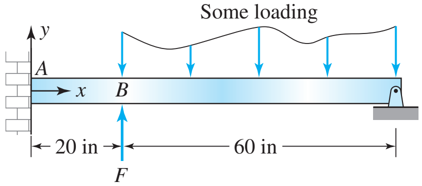


Figure P7.32

7.33 In Figure P7.33, the displacement in the y direction in section AB, is given by $v_1(x) = -3(x^4 - 20x^3)(10^{-6})$ in. and in BC by $v_2(x) = -8(x^2 - 100x + 1600)(10^{-3})$ in. If the bending rigidity is 135×10^6 lb·in.², determine: (a) the reaction force at B and the applied moment M_B; (b) the reactions at the wall at A.

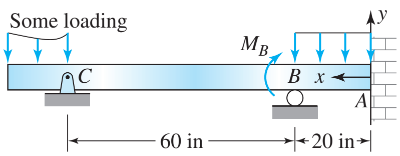


Figure P7.33

7.34 For the beam shown in Figure P7.34, determine the elastic curve and the reaction(s) at A in terms of E, I, P, w, and x.

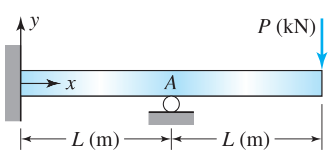


Figure P7.34

7.35 For the beam shown in Figure P7.35, determine the elastic curve and the reaction(s) at A in terms of E, I, P, w, and x.

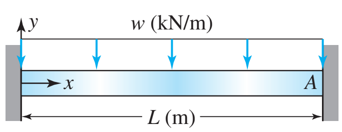


Figure P7.35

7.36 For the beam shown in Figure P7.36, determine the slope at $x = L$ and the reaction moment at the left wall in terms of E, I, w, and L.

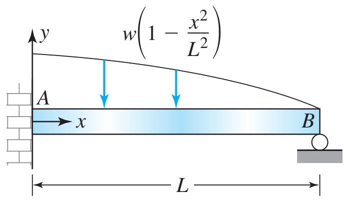


Figure P7.36

7.37 For the beam shown in Figure P7.37, determine the deflection and the moment reaction at $x = L$ in terms of E, I, w, and L.

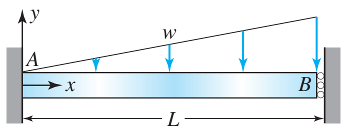


Figure P7.37

7.38 For the beam shown in Figure P7.38, determine the deflection and the slope at $x = L$ in terms of E, I, w, and L.

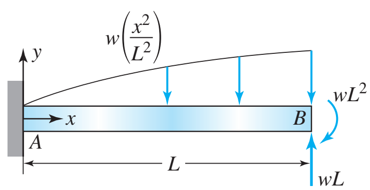


Figure P7.38

7.39 For the beam and loading shown in Figure P7.39, determine the deflection and slope at $x = L$ in terms of E, I, w, and L.

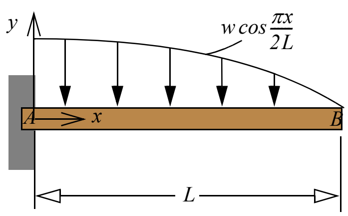


Figure P7.39

7.40 For the beam and loading shown in Figure P7.40, determine the slope at x = L and the reaction moment at the left wall in terms of E, I, w, and L.

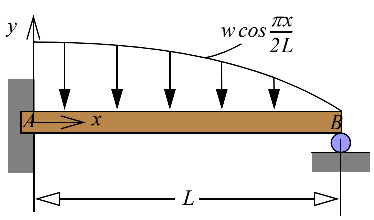


Figure P7.40

7.41 For the beam and loading shown in Figure P7.41, determine the maximum deflection in terms of E, I, w, and L.

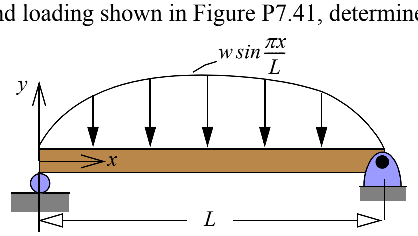


Figure P7.41

7.42 For the beam and loading shown in Figure P7.42, determine the maximum deflection in terms of E, I, w, and L.

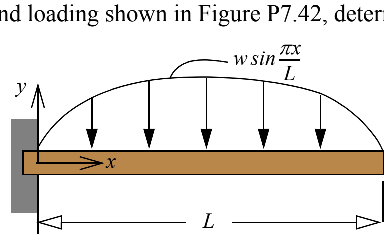


Figure P7.42

7.43 A cantilever beam under uniform load has a spring with a stiffness k attached to it at point A, as shown in Figure P7.43. The spring constant in terms of stiffness of the beam is written as $k = \alpha EI/L^3$, where α is a proportionality factor. Determine the compression of the spring in terms of α, w, E, I, and L.

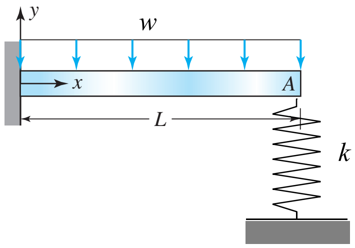


Figure P7.43

7.44 A linear spring that has a spring constant K is attached at the end of a beam, as shown in Figure P7.44. In terms of w, E, I, L, and K, write the boundary-value problem but do not integrate or solve.

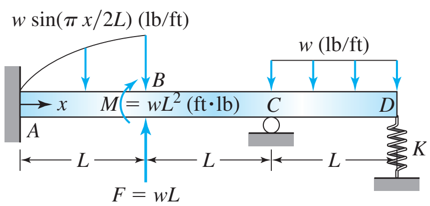
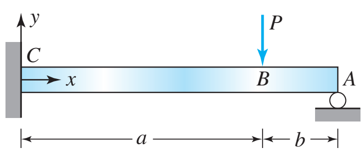


Figure P7.44

Historical problems

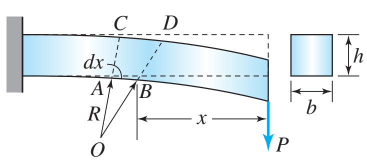
7.45 The beam and loading shown in Figure P7.45 was the first statically indeterminate beam for which a solution was obtained by Navier. Verify that Navier's solution for the reaction at A is given by the equation below.



$$R_A = \frac{Pa^2(3L-a)}{2L^3} \qquad \text{where } L = a + b$$

Figure P7.45

7.46 Jacob Bernoulli incorrectly assumed that the neutral axis was tangent to the concave side of the curve in Figure P7.46 and obtained the equation given below. In the equation R is the radius of curvature of the beam at any location x. Derive this equation based on Bernoulli's assumption and show that it is incorrect by a factor of 4. (*Hint:* Follow the process in Section 6.1 and take the moment about point B.)



$$\frac{Ebh^3}{3}\left(\frac{1}{R}\right) = Px$$

Figure P7.46

7.47 Clebsch considered a beam loaded by several concentrated forces P_i placed at a location x_i, as shown in Figure P7.47. He obtained the second-order differential equation between the concentrated forces. By integration he obtained the slope and deflection as given and concluded that all C_i's are equal and all D_i's are equal. Show that his conclusion is correct. For $x_i \le x \le x_{i+1}$,

$$EI\frac{d^2v}{dx^2} = Rx - \sum_{j=1}^{i} P_j(x - x_j) \qquad EI\frac{dv}{dx} = R\frac{x^2}{2} - \sum_{i=1}^{i} P_i\frac{(x-x_i)^2}{2} + C_i \qquad EIv = R\frac{x^3}{6} - \sum_{i=1}^{i} P_i\frac{(x-x_i)^3}{6} + C_i x + D_i$$

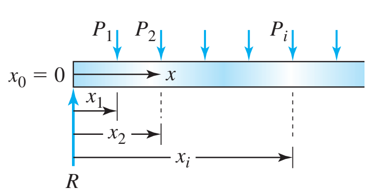
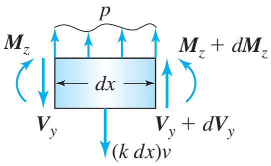


Figure P7.47

Stretch Yourself

7.48 A beam resting on an elastic foundation has a distributed spring force that depends on the deflections at a point acting as shown in Figure P7.48. Show that the differential equation governing the deflection of the beam is given by Equation (7.8), where k is the foundation modulus, that is, spring constant per unit length.



$$\frac{d^2}{dx^2}\left(EI_{zz}\frac{d^2v}{dx^2}\right) + kv = p \tag{7.8}$$

Figure P7.48 Elastic foundation effect.

7.49 To account for shear, the assumption of planes remaining perpendicular to the axis of the beam (Assumption 3 in Section 6.2) is dropped, and it is assumed that the plane rotates by the angle ψ from the vertical. This yields the following displacement equations:

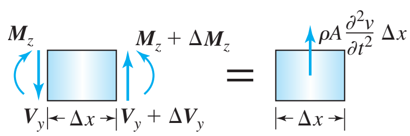
$$u = -y\psi(x) \qquad v = v(x)$$

The rest of the derivation[1] is as before. Show that the following equations apply:

$$\frac{d}{dx}\left[GA\left(\frac{dv}{dx} - \psi\right)\right] = -p \qquad \frac{d}{dx}\left(EI_{zz}\frac{d\psi}{dx}\right) = -GA\left(\frac{dv}{dx} - \psi\right) \tag{7.9}$$

where A is the cross-sectional area and G is the shear modulus of elasticity. Beams governed by these equations are called *Timoshenko beams*.

7.50 Figure P7.50 shows a differential element of a beam that is free to vibrate, where ρ is the material density, A is the cross-sectional area, and $\partial^2 v/\partial t^2$ is the linear acceleration. Show that the dynamic equilibrium is given by Equation (7.10).



$$\frac{\partial^2 v}{\partial t^2} + c^2\frac{\partial^4 v}{\partial x^4} = 0 \qquad \text{where } c = \sqrt{EI_{zz}/\rho A}. \tag{7.10}$$

Figure P7.50 Dynamic equilibrium.

7.51 Show by substitution that the following solution satisfies Equation (7.10):

$$v(x,t) = G(x)H(t) \qquad G(x) = A\cos\omega x + B\sin\omega x + C\cosh\omega x + D\sinh\omega x \qquad H(t) = E\cos(c\omega^2)t + D\sin(c\omega^2)t$$

7.52 Show by substitution that the following deflection solution satisfies the fourth order boundary value problem of the cantilever beam shown in Figure P7.52.

$$v(x) = \frac{1}{6EI}\left[R_A x^3 + 3M_A x^2 + \int_0^x (x - x_1)^3 p(x_1)\,dx_1\right] \qquad \text{where } R_A = -\int_0^L p(x_1)\,dx_1 \text{ and } M_A = \int_0^L x_1 p(x_1)\,dx_1. \tag{7.11}$$

Computer problems

7.53 Table P7.53 shows the value of distributed load at several point along the axis of a 10 ft long rectangular beam. Determine the slope and deflection at the free end using. Use modulus of elasticity as 2000 ksi.

TABLE P7.53 **Data in Problem 7.53**

x (ft)	$p(x)$ (lb/ft)	x (ft)	$p(x)$ (lb/ft)
0	275	6	377
1	348	7	316
2	398	8	233
3	426	9	128
4	432	10	0
5	416		

[1]Use Equations (2.12a) and (2.12d) to get ε_{xx} and γ_{xy}. Use Hooke's law, the static equivalency equations [Equations (6.1) and (6.13)], and the equilibrium equations [Equations (6.17) and (6.18)].

7.54 For the beam and loading given in Problem 7.53, determine the slope and deflection at the free end in the following manner. First represent the distributed load by $p(x) = a + bx + cx^2$ and, using the data in Table P7.53, determine constants *a*, *b*, and *c* by the least-squares method. Then using fourth-order differential equations solve the boundary-value problem. Use the modulus of elasticity as 2000 ksi.

7.55 Table P7.55 shows the measured radii of a solid tapered circular beam at several points along the axis, as shown in Figure P7.55. The beam is made of aluminum (E = 28 GPa) and has a length of 1.5 m. Determine the slope and deflection at point *B*.

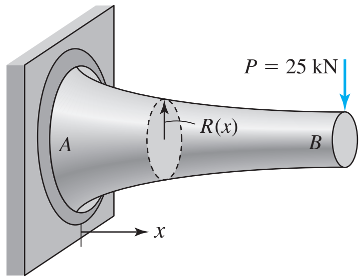


Figure P7.55

TABLE P7.55 **Data for Problem 7.55**

x (m)	*R*(*x*) (mm)	*x* (m)	*R*(*x*) (mm)
0.0	100.6	0.8	60.1
0.1	92.7	0.9	60.3
0.2	82.6	1.0	59.1
0.3	79.6	1.1	54.0
0.4	75.9	1.2	54.8
0.5	68.8	1.3	54.1
0.6	68.0	1.4	49.4
0.7	65.9	1.5	50.6

7.56 Let the radius of the tapered beam in Problem 7.55 be represented by the equation $R(x) = a + bx$. Using the data in Table P7.55, determine constants *a* and *b* by the least-squares method and then find the slope and deflection at point *B* by analytical integration.

MoM in Action: Skyscrapers

A skyscraper can be a monument to the builder's pride or a *literal* monument, designed to attract tourists and tenants to the city's, the country's, or the world's tallest building. When John Roscoe, in 1930, wanted a taller building than Walter Chrysler's, he pushed for his own, completed just one year after the Chrysler building More than 40 years later, his Empire State Building (Figure 7.25*a*). was still the world's tallest building, at 1250 ft. Even today, it is surely the most famous skyscraper ever. New construction is also driven by the same social forces as those behind the boom in Chicago, New York, and London at the end of 19th century. Businesses then and now want to be near a city's commercial center and emerging economies are seeing a movement of the population from villages to cities. As this edition goes to press, the tallest building is Taipei 101 in Taiwan (Figure 7.25*b*). Built in 2003, it stands 1671 feet tall.

(a) (b) (c)

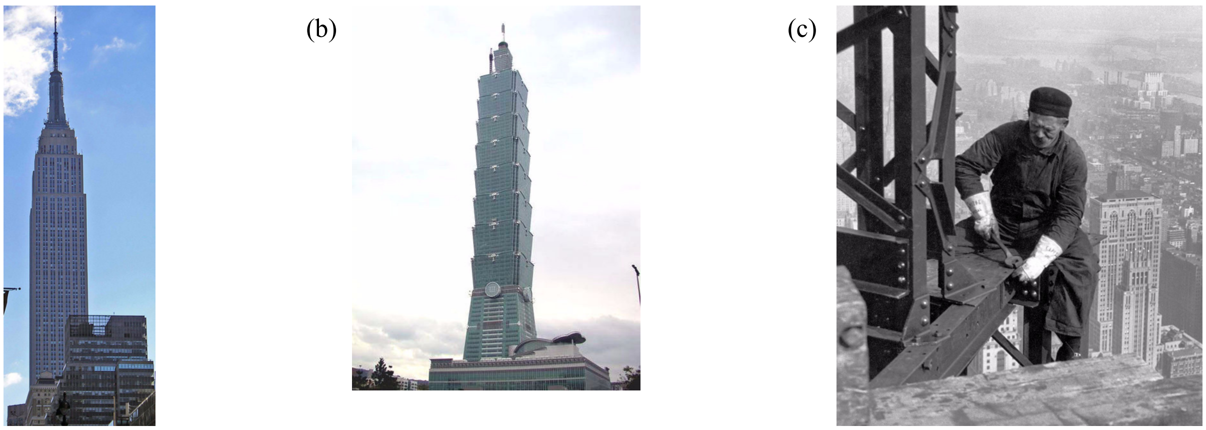


Figure 7.25 (a) Empire State in New York. (b) Taipei 101 in Taipei, Taiwan. (c) Joint construction.

Social forces, then, have pushed skyscrapers higher and higher, but technological advances have made that possible. Early high-rise buildings had a pyramid design: the building cross-section decreased with height to avoid excessive stresses at the bottom. The height of these building was limited by the strength of masonry materials and by the difficulty of getting water to higher stories. Besides, renters did not want to climb too many stairs! With the advent of steel beams, reinforced concrete, glass, electric water pumps, and elevators, however, the human imagination was freed to build tall. If one thinks of a high-rise building as an axial column, then skyscrapers are like cantilevered beams subject to bending loads in the wind. A proper variation of both axial and bending rigidity with height is important in design. Skyscrapers must be strong enough to withstand hurricane winds in excess of 140 mph. With an increase in height, too, the bearing stresses at the base increase, often requiring digging deep to bedrock.

In addition to the stresses, the deflection of a skyscraper increases with height. By welding and bolting the horizontal girders to steel columns, the rigidity of the joint (Figure 7.25*c*) is increased, which helps reduce the sway. Skyscraper designs often have columns on the outer perimeter, which are connected to the central core columns. The outer columns act like flanges to resist most of the wind load, while the inner columns carry most of the weight. In modern skyscrapers, computer-controlled masses of hundreds of tons, called *tuned mass dampers*, move to counter the building sway. Today skyscrapers are also designed to move *with* earthquakes rather then stress the building frames.

The terrorist attack on World Trade Center (see page 525) has highlighted the need for better fireproofing of steel beams, and technology is once more providing the solution. But terrorist acts do not deter the human spirit, which like the skyscrapers themselves still soars. In the sands of United Arab Emirates, the next tallest skyscraper is rising. Called *Burj Dubai*, or the *Dubai tower*, it will be twice the height of the Empire State Building.

7.3* SUPERPOSITION

The assumptions and limitations that were imposed in deriving the simplest theory for beam bending ensured that we have a linear theory. As a consequence, the differential equations governing beam deflection, Equations (7.1) and (7.5), are linear differential equations, and hence the principle of superposition can be applied to beam deflection.

The leftmost beam in Figure 7.26 is loaded with a uniformly distributed load w and a concentrated load P_1. The superposition principle says that the deflection of a beam with uniform load w and point force P_1 is equal to the sum of the deflections calculated by considering each load separately, as shown on the right two beams in Figure 7.26. Although the example in Figure 7.26 demonstrates the principle of superposition, there is no intrinsic gain in calculating the deflection of each load separately and adding to find the final answer. But if the solutions to basic cases are tabulated, as in Table C.3, then the principle of superposition becomes a very useful tool to obtain results quickly. Thus the maximum deflection of the beam on the left can be found using the results of cases 1 and 3 in Table C.3. Comparing the loading of the two beams in Figure 7.26 to those shown for cases 1 and 3, we note that $P = -P_1 = -wL$ and $p_0 = -w$, $a = L$, and $b = 0$. Substituting these values into v_{max} given in Table C.3 and adding, we obtain

$$v_{max} = -\frac{(wL)L^3}{3EI} - \frac{wL^4}{8EI} = -\frac{11wL^4}{24EI} \tag{7.12.a}$$

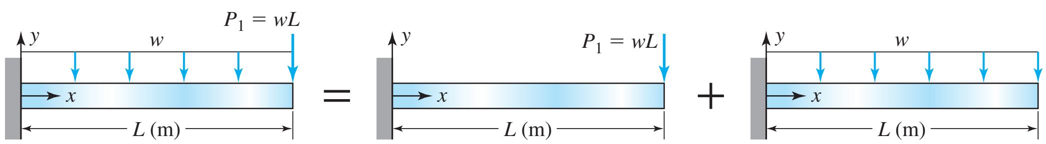


Figure 7.26 Example of superposition principle.

Another very useful application of superposition is the deflection of statically indeterminate beams. Consider a beam built in at one end and simply supported at the other end with a uniformly distributed load, as shown in Figure 7.27. The support at A can be replaced by a reaction force, and once more the total loading can be shown as the sum of two individual loads, as shown at right in Figure 7.27. Comparing the loading of the two beams in Figure 7.26 to those shown for cases 1 and 3 in Table C.3, we note that $P = R_A$, $p_0 = -w$, $a = L$, and $b = 0$. v_{max} is at point A in both cases. Substituting these values into v_{max} given in Table C.3 and adding, we obtain

$$v_A = \frac{R_A L^3}{3EI} - \frac{wL^4}{8EI} \tag{7.12.b}$$

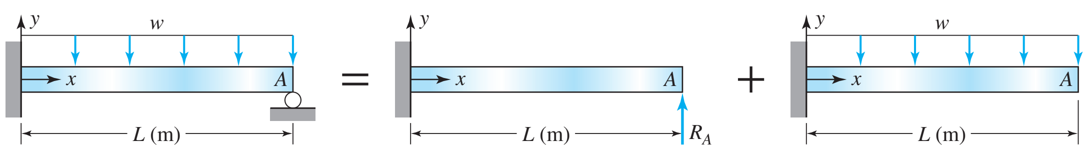


Figure 7.27 Example of use of superposition principle in solving statically indeterminate beam deflection.

But the deflection at A must be zero in the original beam. Thus we can solve for the reaction force as $R_A = 3wL/8EI$. Now the solution of $v(x)$ given in Table C.3 can be superposed to obtain

$$v(x) = \frac{R_A x^2}{6EI}(3L - x) + \frac{(-w)x^2}{24EI}(x^2 - 4Lx + 6L^2) \tag{7.12.c}$$

Substituting for R_A and simplifying, the solution for the elastic curve is

$$v(x) = \frac{wx^2(-2x^2 + 5Lx - 3L^2)}{48EI} \tag{7.12.d}$$

EXAMPLE 7.8

For the beam shown in Figure 7.28, using the principle of superposition and Table C.3, determine (a) the reactions at A; (b) the maximum deflection.

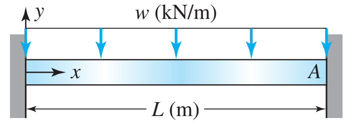


Figure 7.28 Beam in Example 7.8.

PLAN

(a) The wall at A can be replaced by a force reaction R_A and a moment reaction M_A. Thus the beam would be a cantilever beam with a uniformly distributed load, a point force at the end, and a point moment at the end, corresponding to the first three cases in Table C.3. Superposing the slope and deflection values from Table C.3 and equating the result to zero would generate two equations in the two unknowns R_A and M_A which give the reactions at A. (b) From the symmetry of the problem, we can conclude that the maximum deflection will occur at the center. Substituting $x = L/2$ in the elastic curve equation of Table C.3 and adding the results, we can find the maximum deflection of the beam.

SOLUTION

(a) The right wall at A can be replaced by a reaction force and a reaction moment, as shown at left in Figure 7.29. The total loading on the beam can be considered as the sum of the three loadings shown at right in Figure 7.29.

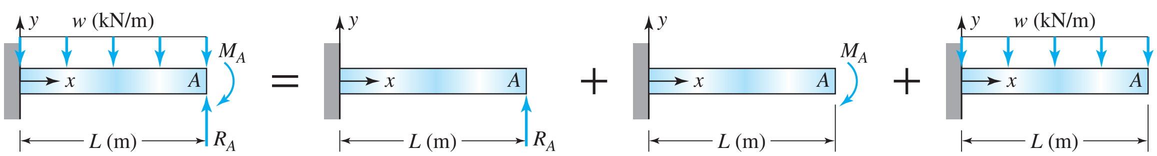


Figure 7.29 Superposition of three loadings in Example 7.8.

Comparing the three beam loadings in Figure 7.29 to that shown for cases 1 through 3 in Table C.3, we obtain $P = R_A$, $M = -M_A$, and $p = -w$, $a = L$, and $b = 0$. Noting that v_{max} and θ_{max} shown in Table C.3 for the cantilever beam occur at point A, we can substitute the load values and superpose to obtain the deflection v_A and the slope at θ_A. Noting that at the wall at A the deflection v_A and the slope at θ_A must be zero, we obtain two simultaneous equations in R_A and M_A,

$$v_A = \frac{R_A L^3}{3EI} + \frac{(-M_A)L^2}{2EI} + \frac{(-w)L^4}{8EI} = 0 \qquad \text{or} \qquad 8R_A L - 12M_A = 3wL^2 \tag{E1}$$

$$\theta_A = \frac{R_A L^2}{2EI} + \frac{(-M_A)L}{EI} + \frac{(-w)L^3}{6EI} = 0 \qquad \text{or} \qquad 3R_A L - 6M_A = wL^2 \tag{E2}$$

Equations (E1) and (E2) can be solved to obtain R_A and M_A.

$$\textbf{ANS.} \qquad R_A = \frac{wL}{2} \qquad M_A = \frac{wL^2}{12} \tag{E3}$$

(b) The maximum deflection would occur at the center of the beam. Substituting $x = L/2$, $P = R_A = wL/2$, $M = -M_A = -wL/12$, and $p = -w$ in the equation of the elastic curve for cases 1 through 3 in Table C.3 and superposing the solution, we obtain

$$v_{max} = v\left(\frac{L}{2}\right) = \frac{(wL/2)(L/2)^2}{6EI}\left(3L - \frac{L}{2}\right) + \frac{-(wL^2/12)(L/2)^2}{2EI} + \frac{(-w)(L/2)^2}{24EI}\left[\left(\frac{L}{2}\right)^2 - 4L\left(\frac{L}{2}\right) + 6L^2\right] \qquad \text{or} \tag{E4}$$

$$v_{max} = \frac{5wL^4}{96EI} - \frac{wL^4}{96EI} - \frac{17wL^4}{384EI} \tag{E5}$$

$$\textbf{ANS.} \qquad v_{max} = -\frac{wL^4}{384EI}$$

COMMENTS

1. All terms in Equations (E1) and (E2) have the same dimension, as they should. If this were not the case, then we would need to examine the equations obtained using superposition and the subsequent simplifications carefully to ensure dimensional homogeneity.
2. By symmetry we know that the reaction forces at each wall must be equal. Hence the value of the reaction forces should be $wL/2$, as calculated in Equation (E3).

EXAMPLE 7.9

The end of one cantilever beam rests on the end of another cantilever beam, as shown in Figure 7.30. Both beams have length L and bending rigidity EI. Determine the deflection at A and the wall reactions at B and C in terms of w, L, E, and I.

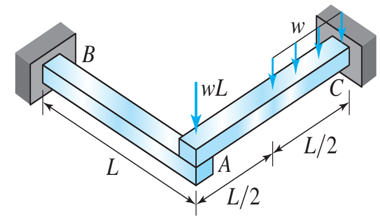


Figure 7.30 Two cantilever beams in Example 7.9.

PLAN

The two beams can be separated by putting an unknown force R_A that is equal but opposite in direction on each beam at point A. From case 1 in Table C.3 for beam AB, the deflection at A can be found in terms of R_A. From cases 1 and 3 for beam AC, the deflection at A can be found by superposition. By equating the deflection at A for the two beams, we can find the force R_A. (a) Once R_A is known, the deflection at A is found from the equation written for beam AB. (b) The reactions at B and C can be found using equilibrium equations on each beam's free-body diagram.

SOLUTION

(a) The assembly of the beams shown in Figure 7.30 can be represented by two beams with a force R_A that acts in equal but opposite directions, as shown in Figure 7.31*a* and *b*. The loading on the beam in Figure 7.31*b* can be represented as the sum of the two loadings shown in Figure 7.31*c* and *d*.

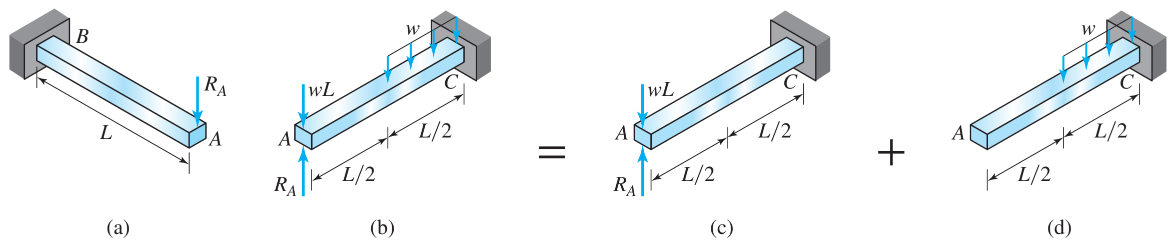


Figure 7.31 Analysis of beam assembly by superposition in Example 7.9.

Comparing the beam of Figure 7.31*a* to that shown in case 1 in Table C.3, we obtain $P = -R_A$, $a = L$, and $b = 0$. Noting that v_{max} in case 1 occurs at A, the deflection at A can be written as

$$v_A = \frac{(-R_A)L^2}{6EI}2L = -\frac{R_A L^3}{3EI} \tag{E1}$$

Comparing the beam in Figure 7.31*c* to that of case 1 in Table C.3, we obtain $P = R_A - wL$, $a = L$, and $b = 0$. Comparing the beam in Figure 7.31*d* to that of case 3 in Table C.3, we obtain $p_0 = -w$, $a = L/2$, and $b = L/2$. Since v_{max} for both cases occurs at A, by superposition the deflection at A can be written as

$$v_A = \frac{(R_A - wL)L^2}{6EI}2L + \frac{(-w)(L/2)^3(3L/2 + 4L/2)}{24EI} = \frac{(R_A - wL)L^3}{3EI} - \frac{7wL^4}{384EI} \tag{E2}$$

Equating Equations (E1) and (E2), give the reaction R_A:

$$-\frac{R_A L^3}{3EI} = \frac{(R_A - wL)L^3}{3EI} - \frac{7wL^4}{384EI} \quad \text{or} \quad R_A = \frac{135wL}{256} \tag{E3}$$

Substituting Equation (E3) into Equation (E1), we obtain the deflection at A.

ANS. $v_A = -\dfrac{45wL^4}{256EI}$

(b) The reactions at the wall can be found from the free-body diagrams of each beam, as shown in Figure 7.32. By equilibrium of forces in the y direction and the moments about B in Figure 7.32*a*, the reactions at B can be found,

$$R_B = R_A \qquad M_B = R_A L \tag{E4}$$

ANS. $R_B = \dfrac{135wL}{256}$ $\qquad M_B = \dfrac{135wL^2}{256}$

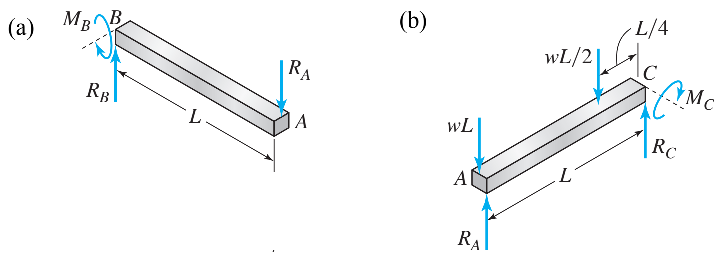


Figure 7.32 Free-body diagrams in Example 7.9.

By equilibrium of forces in the y direction and the moments about C in Figure 7.32b, the reactions at C can be found,

$$R_C = wL + \frac{wL}{2} - R_A = \frac{249wL}{256} \tag{E5}$$

$$M_C = wL(L) + \frac{wL}{2}\left(\frac{L}{4}\right) + R_A L = \frac{153wL^2}{256} \tag{E6}$$

ANS. $R_C = \dfrac{249wL}{256}$ $\quad M_C = \dfrac{153wL^2}{256}$

COMMENT

1. This example demonstrates how the principle of superposition can significantly simplify the analysis and design of structures. Handbooks now document an extensive number of cases for which beam deflections are known. These apply to a wide variety of beam assemblies. But to develop a list of formulas (as in Table C.3) requires a knowledge of the methods described in Sections 7.1 and 7.2.

7.4* DEFLECTION BY DISCONTINUITY FUNCTIONS

Thus far, we have used different functions to represent the distributed load p_y, or moment $\boldsymbol{M}_z$, for different parts of the beam. We then had to determine the integration constants that satisfy the continuity conditions and equilibrium conditions at the junctions x_j. These tedious and algebraically intensive tasks, may be unavoidable for a complicated distributed loading function. But for many engineering problems, where the distributed loads either are constant or vary linearly, there is an alternative method that avoids the algebraic tedium. The method is based on the concept of *discontinuity functions*.

7.4.1 Discontinuity Functions

Consider a distributed load p and an equivalent load $P = p\varepsilon$, as shown in Figure 7.33. Suppose we now let the intensity of the distributed load increase continuously to infinity. At the same time, we decrease the length over which the distributed force is applied to zero so that the area $p\varepsilon$ remains a finite quantity. We then obtain a concentrated force P applied at $x = a$. Mathematically,

$$P = \lim_{p \to \infty} \lim_{\varepsilon \to 0} (p\varepsilon)$$

Rather than write the limit operations, we can represent a concentrated force with, $P\langle x - a\rangle^{-1}$.

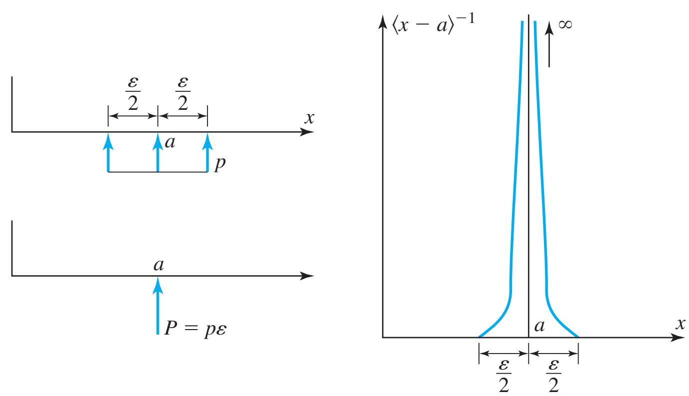


Figure 7.33 Delta function.

The function $\langle x-a\rangle^{-1}$ is called the *Dirac delta function,* or **delta function**. The delta function is zero except in an infinitesimal region near a. As x tends toward a, the delta function tends to infinity, but the area under the function is equal to 1. Mathematically, the delta function is defined as

$$\boxed{\langle x-a\rangle^{-1} = \begin{cases} 0, & x \neq a \\ \infty, & x \to a \end{cases} \qquad \int_{a-\varepsilon}^{a+\varepsilon} \langle x-a\rangle^{-1}\, dx = 1} \tag{7.13}$$

Now consider the following integral of the delta function:

$$\int_{-\infty}^{x} \langle x-a\rangle^{-1}\, dx$$

The lower limit of minus infinity emphasizes that the point is before a. If $x < a$, then in the interval of integration, the delta function is zero at all points; hence the integral value is zero. If $x > a$, then the integral can be written as the sum of three integrals,

$$\int_{-\infty}^{a-\varepsilon} \langle x-a\rangle^{-1}\, dx + \int_{a-\varepsilon}^{a+\varepsilon} \langle x-a\rangle^{-1}\, dx + \int_{a+\varepsilon}^{x} \langle x-a\rangle^{-1}\, dx$$

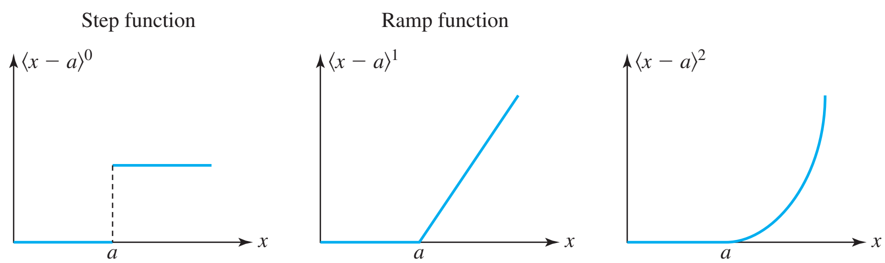


Figure 7.34 Discontinuity functions.

The first and third integrals are zero because the delta function is zero at all points in the interval of integration, whereas the second integral is equal to 1 as per Equation (7.13). Thus the integral $\int_{-\infty}^{x} \langle x-a\rangle^{-1}\, dx$ is zero before a and one after a. It is called the **step function** as shown in Figure 7.34 and is represented by the notation $\langle x-a\rangle^{0}$.

$$\langle x-a\rangle^{0} = \int_{-\infty}^{x} \langle x-a\rangle^{-1}\, dx = \begin{cases} 0, & x < a \\ 1, & x > a \end{cases} \tag{7.14}$$

Now consider the integral of the step function,

$$\int_{-\infty}^{x} \langle x-a\rangle^{0}\, dx$$

If $x < a$, then in the interval of integration the step function is zero at all points. Hence the integral value is zero. If $x > a$, then we can write the integral as the sum of two integrals,

$$\int_{-\infty}^{a} \langle x-a\rangle^{0}\, dx + \int_{a}^{x} \langle x-a\rangle^{0}\, dx$$

The first integral is zero because the step function is zero at all points in the interval of integration, whereas the second integral value is $x - a$. The integral $\int_{-\infty}^{x} \langle x-a\rangle^{0}\, dx$ is called the **ramp function**. It is represented by the notation $\langle x-a\rangle^{1}$ and is shown in Figure 7.34. Proceeding in this manner we can define an entire class of functions, which are represented mathematically as follows:

$$\boxed{\langle x-a\rangle^{n} = \begin{cases} 0, & x \le a \\ (x-a)^{n}, & x > a \end{cases}} \tag{7.15}$$

We can also generate the following integral formula from Equation (7.15):

$$\int_{-\infty}^{x} \langle x-a\rangle^{n}\,dx = \frac{\langle x-a\rangle^{n+1}}{n+1}, \qquad n \ge 0 \tag{7.16}$$

We define one more function, called the **doublet function**. It is represented by the notation $\langle x-a\rangle^{-2}$ and is defined mathematically as

$$\langle x-a\rangle^{-2} = \begin{cases} 0, & x \ne a \\ \infty, & x \to a \end{cases} \qquad \int_{-\infty}^{x} \langle x-a\rangle^{-2}\,dx = \langle x-a\rangle^{-1} \tag{7.17}$$

The delta function $\langle x-a\rangle^{-1}$ and the doublet function $\langle x-a\rangle^{-2}$ become infinite at $x = a$, that is, they are singular at $x = a$ and are referred to as **singularity functions**. The entire class of functions $\langle x-a\rangle^{n}$ for positive and negative n are called **discontinuity functions**.

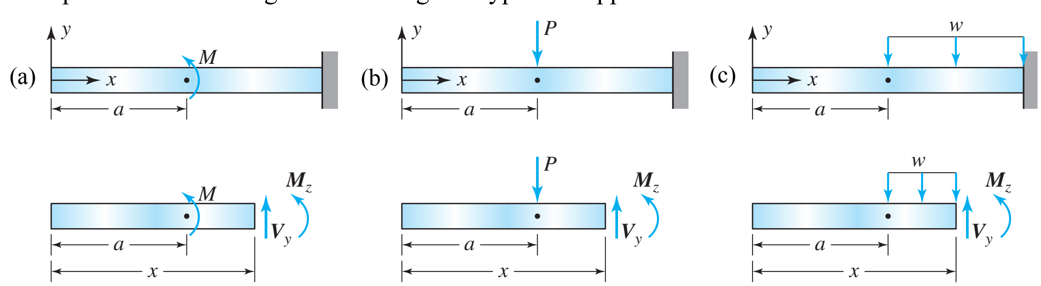
The discontinuity functions are zero if the argument is negative. By differentiating Equations (7.14), (7.16), and (7.17) we can obtain the following formulas:

$$\frac{d\langle x-a\rangle^{-1}}{dx} = \langle x-a\rangle^{-2} \qquad \frac{d\langle x-a\rangle^{0}}{dx} = \langle x-a\rangle^{-1} \tag{7.18}$$

$$\frac{d\langle x-a\rangle^{n}}{dx} = n\langle x-a\rangle^{n-1}, \qquad n \ge 1 \tag{7.19}$$

7.4.2 Use of Discontinuity Functions

Before proceeding to develop a method for solving for the elastic curve using discontinuity functions, we discuss the process by which the internal moment $\boldsymbol{M}_z$ and the distributed force p_y can be written using the discontinuity functions. We will develop the procedure using a simple example of a cantilever beam subject to different types of loading, as shown in Figure 7.35. Then we will generalize the procedure to more general loading and types of support.



$$\boldsymbol{M}_z = \begin{cases} 0 & x < a \\ -M & x > a \end{cases} \qquad \boldsymbol{M}_z = \begin{cases} 0 & x < a \\ -P(x-a) & x > a \end{cases} \qquad \boldsymbol{M}_z = \begin{cases} 0 & x < a \\ -\dfrac{w(x-a)^2}{2} & x > a \end{cases}$$

$$\boldsymbol{M}_z = -M\langle x-a\rangle^{0} \qquad \boldsymbol{M}_z = -P\langle x-a\rangle^{1} \qquad \boldsymbol{M}_z = -w\frac{\langle x-a\rangle^{2}}{2}$$

$$p = -M\langle x-a\rangle^{-2} \qquad p = -P\langle x-a\rangle^{-1} \qquad p = -w\langle x-a\rangle^{0}$$

Figure 7.35 Use of discontinuity functions.

When we make an imaginary cut before $x = a$ in the cantilever beams shown in Figure 7.35, the internal moment $\boldsymbol{M}_z$ will be zero. If the imaginary cut is made after $x = a$, then the internal moment $\boldsymbol{M}_z$ will not be zero and can be determined using a free-body diagram. Once the moment expression is known, then it can be rewritten using the discontinuity functions. This moment expression can be used to find displacement using the second-order differential equation, Equation (7.1). However, if the fourth-order differential equation, Equation (7.5), has to be solved, then the expression of the distributed force p_y is needed. Now the distributed force can be obtained from the moment expression using the identity that is obtained by substituting Equation (6.18) into Equation (6.17), or $d^2\boldsymbol{M}_z/dx^2 = p_y$. By using Equation (7.19) we can obtain the distributed force expression from the moment expression, as shown in Figure 7.35.

If the distributed load is as shown in Figure 7.35*c*, then the expression for it can be obtained directly, without the free-body diagram, and the moment expression can be obtained by integrating twice. For the concentrated force and moment also, it is not difficult to recognize the type of discontinuity function that will be used in the representation. The difficulty lies in obtaining the correct sign in the expression for the internal moment $\boldsymbol{M_z}$. We shall overcome this problem by using a template to guide us.

A template is created by making an imaginary cut beyond the applied load. On the imaginary cut the internal moment is drawn according to the sign convention discussed in Section 6.2.6. A moment equilibrium equation is written. If the applied load is in the assumed direction on the template, then the sign used is the sign in the moment equilibrium equation. If the direction of the applied load is opposite to that on the template, then the sign in the equilibrium equation is changed. The beams shown in Figure 7.35 are like templates for the given coordinate systems.

EXAMPLE 7.10

Write the moment and distributed force expressions using discontinuity functions for the three templates shown in Figure 7.36.

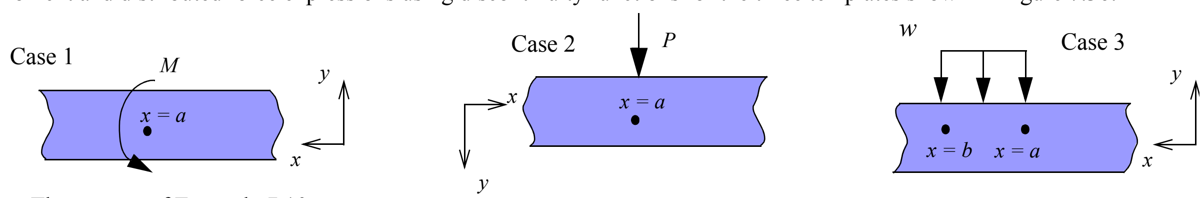


Figure 7.36 Three cases of Example 7.10.

PLAN

For cases 1 and 2 we can make an imaginary cut after $x = a$ and draw the shear force and bending moment according to the sign convention in Section 6.2.6. By equilibrium we can obtain the moment expression and rewrite it using discontinuity functions. By differentiating twice, we can obtain the distributed force expression. For case 3 we can write the expression for the distributed force using discontinuity functions and integrate twice to obtain the moment expression.

SOLUTION

Case 1: We make an imaginary cut at $x > a$ and draw the free-body diagram using the sign convention in Section 6.2.6 as shown in Figure 7.37*a*. By equilibrium we obtain

$$\boldsymbol{M_z} = M \tag{E7}$$

is valid only after $x > a$. Using the step function we can write the moment expression, and by differentiating twice as per Equation (7.19) we obtain our result.

ANS. $\boldsymbol{M_z} = M\langle x-a\rangle^0 \qquad p_y = M\langle x-a\rangle^{-2}$

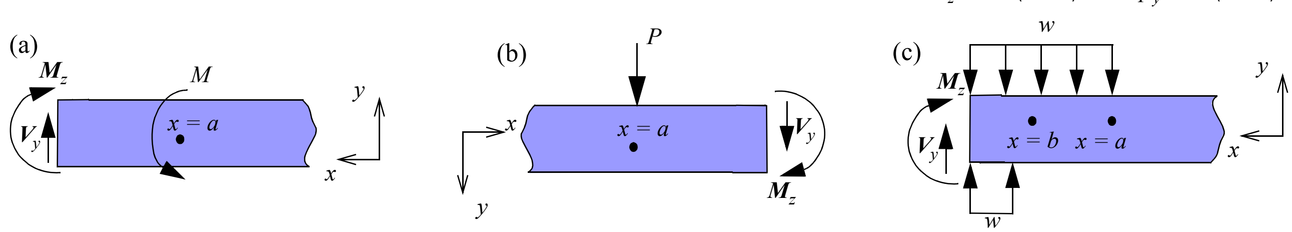


Figure 7.37 (a) Case 1, (b) Case 2, (c) Case 3 in Example 7.10.

Case 2: We make an imaginary cut at $x > a$ and draw the free-body diagram using the sign convention in Section 6.2.6 as shown in Figure 7.37*b*. By equilibrium we obtain

$$\boldsymbol{M_z} = P(x-a) \tag{E8}$$

is valid only after $x > a$. Using the ramp function we can write the moment expression, and by differentiating twice as per Equation (7.19) we obtain our result.

ANS. $\boldsymbol{M_z} = P\langle x-a\rangle^1 \qquad p_y = P\langle x-a\rangle^{-1}$

Case 3: The distributed force is in the negative y direction. Its start can be represented by the step function at $x = a$. The end of the distributed force can also be represented by a step function using a sign opposite to that used at the start as shown in Figure 7.37*b*.

ANS. $$p_y = -w\langle x-a\rangle^0 + w\langle x-b\rangle^0 \tag{E9}$$

Integrating Equation (E9) twice and using Equation (7.16), we obtain the moment expression

ANS. $$M_z = -\frac{w}{2}\langle x-a\rangle^2 + \frac{w}{2}\langle x-b\rangle^2 \qquad \text{(E10)}$$

COMMENTS

1. The three cases shown could be part of a beam with more complex loading. But the contribution for each of the loads would be calculated as shown in the example.
2. In obtaining Equation (E10) we did not yet write integration constants. When we integrate for displacements, we will determine these from boundary conditions.
3. In case 3 we did not have to draw the free-body diagram. This is an advantage when the distributed load changes character over the length of the beam. Even for statically determinate beams, it may be advantageous to start with the fourth-order, rather than the second-order differential equation.

EXAMPLE 7.11

Using discontinuity functions, determine the equation of the elastic curve in terms of E, I, L, P, and x for the beam shown in Figure 7.38.

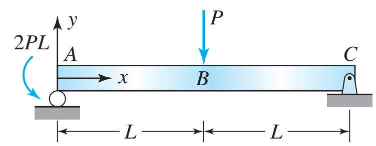


Figure 7.38 Beam and loading in Example 7.11.

PLAN

Two templates can be created, one for an applied moment and one for the applied force. With the templates as a guide, the moment expression in terms of discontinuity functions can be written. The second-order differential equation, Equation (7.1), can be written and solved using the zero deflection boundary conditions at A and C to obtain the elastic curve.

SOLUTION

Figure 7.39 shows two templates. By equilibrium, the moment expressions for the two templates can be written

$$M_z = M\langle x-a\rangle^0 \qquad M_z = F\langle x-a\rangle^1 \qquad \text{(E1)}$$

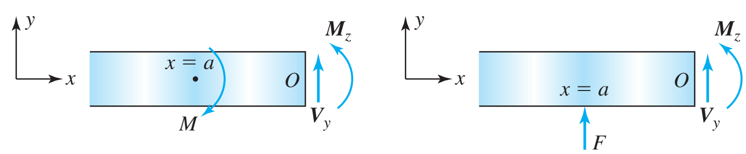


Figure 7.39 Templates for Example 7.11.

Figure 7.40 shows the free-body diagram of the beam. By equilibrium of moment at C, the reaction at A can be found as $R_A = 3P/2$. We can write the moment expressions using the templates in Figure 7.39 to guide us. The reaction force is in the same direction as the force in the template. Hence the term in Equation (E2) will have the same sign as shown in the template equation.
The applied moment at point A has an opposite direction to that shown in the template in Figure 7.39. Hence the term in the moment expression in Equation (E2) will have a negative sign to that shown in the template equation. The force P at B has an opposite sign to that shown on the template, and hence the term in the moment expression will have a negative sign, as shown in Equation (E2).

$$M_z = \frac{3P}{2}\langle x\rangle^1 - 2PL\langle x\rangle^0 - P\langle x-L\rangle^1 \qquad \text{(E2)}$$

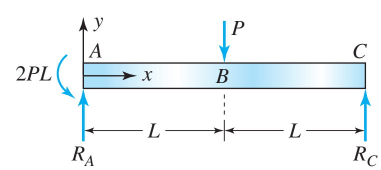


Figure 7.40 Free-body diagram in Example 7.11.

Substituting Equation (E2) into Equation (7.1) and writing the zero deflection conditions at A and C, we obtain the boundary-value problem:

- **Differential equation:**

$$EI_{zz}\frac{d^2 v}{dx^2} = \frac{3P}{2}\langle x\rangle^1 - 2PL\langle x\rangle^0 - P\langle x-L\rangle^1, \qquad 0 \le x < 2L \qquad \text{(E3)}$$

- **Boundary conditions:**

$$v(0) = 0 \tag{E4}$$

$$v(2L) = 0 \tag{E5}$$

Integrating Equation (E3) twice using Equation (7.16), we obtain

$$EI_{zz}\frac{dv}{dx} = \frac{3P}{4}\langle x\rangle^2 - 2PL\langle x\rangle^1 - \frac{P}{2}\langle x-L\rangle^2 + c_1 \tag{E6}$$

$$EI_{zz}v = \frac{P}{4}\langle x\rangle^3 - PL\langle x\rangle^2 - \frac{P}{6}\langle x-L\rangle^3 + c_1x + c_2 \tag{E7}$$

Substituting Equation (E7) into Equation (E4), we obtain the constant c_2:

$$\frac{P}{4}\langle 0\rangle^3 - PL\langle 0\rangle^2 - \frac{P}{6}\langle -L\rangle^3 + c_2 = 0 \quad \text{or} \quad c_2 = 0 \tag{E8}$$

Substituting Equation (E7) into Equation (E5), we obtain the constant c_1:

$$\frac{P}{4}\langle 2L\rangle^3 - PL\langle 2L\rangle^2 - \frac{P}{6}\langle L\rangle^3 + c_1(2L) = 0 \quad \text{or} \quad c_1 = \frac{13}{12}PL^2 \tag{E9}$$

Substituting Equations (E8) and (E9) into Equation (E7), we obtain the elastic curve.

$$\textbf{ANS.} \quad v(x) = \frac{P}{12EI_{zz}}[3\langle x\rangle^3 - 12L\langle x\rangle^2 - 2\langle x-L\rangle^3 + 13L^2x] \tag{E10}$$

Dimension check: All terms in brackets are dimensionally homogeneous as all have the dimensions of length cubed. But we can also check whether the left-hand side and any one term of the right-hand side have the same dimension,

$$P \rightarrow O(F) \qquad x \rightarrow O(L) \qquad E \rightarrow O\left(\frac{F}{L^2}\right) \qquad I_{zz} \rightarrow O(L^4) \qquad v \rightarrow O(L) \qquad \frac{Px^3}{EI_{zz}} \rightarrow O\left(\frac{FL^3}{(F/L^2)L^4}\right) \rightarrow O(L) \rightarrow \text{checks}$$

COMMENTS

1. Comparing the boundary-value problem in this example with that of Example 7.2, we note the following: (i) There is only one differential equation here representing the two differential equations of Example 7.2. (ii) There are no continuity equations at $x = L$ as there were in Example 7.2. The net impact of these two features is a significant reduction in the algebra in this example compared to the algebra in Example 7.2.
2. Equation (E10) represents the two equations of the elastic curve in Example 7.2. We note that $\langle x-L\rangle^3 = 0$ for $0 \le x < L$. Hence Equation (E10) can be written $v(x) = P(3x^3 - 12Lx^2 + 13L^2x)/12EI_{zz}$, which is same as Equation (E18) in Example 7.2. For $L \le x < 2L$, the term $\langle x-L\rangle^3 = (x-L)^3$. Hence Equation (E10) can be written $v(x) = P[3x^3 - 12Lx^2 + 13L^2x - 2(x-L)^3]/12EI_{zz}$, which is same as Equation (E19) in Example 7.2.

EXAMPLE 7.12

A beam with a bending rigidity $EI = 42{,}000\ \text{N}\cdot\text{m}^2$ is shown in Figure 7.41. Determine: (a) the deflection at point B; (b) the moment and shear force just before and after B.

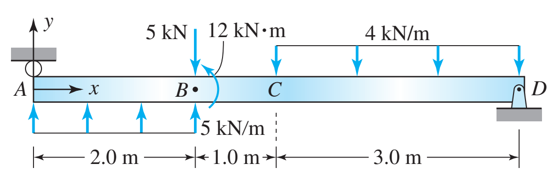


Figure 7.41 Beam and loading in Example 7.12.

PLAN

The coordinate system in this example is the same as in Example 7.11, and hence we can use the templates in Figure 7.39. Differentiating the template equations twice, we obtain the template equation for the distributed forces, and we write the distributed force expression in terms of discontinuity functions. Using Equation (7.5) and the boundary conditions at A and D, we can write the boundary-value problem and solve it to obtain the elastic curve. (a) Substituting $x = 2$ m in the elastic curve, we can obtain the deflection at B. (b) Substituting $x = 2.5$ in the shear force expression, we can obtain the shear force value.

SOLUTION

(a) The templates of Example 7.11 are repeated in Figure 7.42. The moment expression is differentiated twice to obtain the template equations for the distributed force,

$$M_z = M\langle x-a\rangle^0 \qquad M_z = F\langle x-a\rangle^1 \tag{E1}$$

$$p = M\langle x-a\rangle^{-2} \qquad p = F\langle x-a\rangle^{-1} \tag{E2}$$

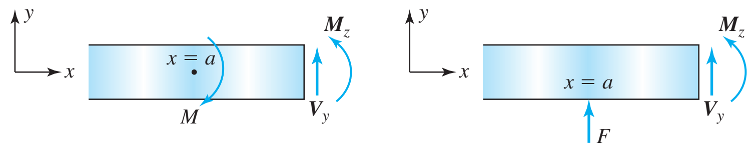


Figure 7.42 Templates for Example 7.12.

We note that the distributed force in segment *AB* is positive, starts at zero, and ends at $x = 2$. The distributed force in segment *CD* is negative, starts at $x = 3$, and is over the rest of the beam. Using the template equations and Figure 7.42, we can write the distributed force expression,

$$p = 5\langle x\rangle^0 - 5\langle x-2\rangle^0 - 4\langle x-3\rangle^0 - 5\langle x-2\rangle^{-1} - 12\langle x-2\rangle^{-2} \tag{E3}$$

Substituting Equation (E3) into Equation (7.5) and writing the boundary conditions, we obtain the boundary-value problem:

- **Differential equation:**

$$\frac{d^2}{dx^2}\left(EI_{zz}\frac{d^2 v}{dx^2}\right) = 5\langle x\rangle^0 - 5\langle x-2\rangle^0 - 4\langle x-3\rangle^0 - 5\langle x-2\rangle^{-1} - 12\langle x-2\rangle^{-2}, \qquad 0 \le x < 6 \tag{E4}$$

- **Boundary conditions:**

$$v(0) = 0 \tag{E5}$$

$$EI_{zz}\frac{d^2 v}{dx^2}(0) = 0 \tag{E6}$$

$$v(6) = 0 \tag{E7}$$

$$EI_{zz}\frac{d^2 v}{dx^2}(6) = 0 \tag{E8}$$

Integrating Equation (E4) twice, we obtain

$$\frac{d}{dx}\left(EI_{zz}\frac{d^2 v}{dx^2}\right) = 5\langle x\rangle^1 - 5\langle x-2\rangle^1 - 4\langle x-3\rangle^1 - 5\langle x-2\rangle^0 - 12\langle x-2\rangle^{-1} + c_1 \tag{E9}$$

$$EI_{zz}\frac{d^2 v}{dx^2} = \frac{5}{2}\langle x\rangle^2 - \frac{5}{2}\langle x-2\rangle^2 - 2\langle x-3\rangle^2 - 5\langle x-2\rangle^1 - 12\langle x-2\rangle^0 + c_1 x + c_2 \tag{E10}$$

Substituting Equation (E10) into Equation (E5), we obtain

$$c_2 = 0 \tag{E11}$$

Substituting Equation (E10) into Equation (E8), we obtain

$$\frac{5}{2}\langle 6\rangle^2 - \frac{5}{2}\langle 4\rangle^2 - 2\langle 3\rangle^2 - 5\langle 4\rangle^1 - 12\langle 4\rangle^0 + c_1(6) = 0 \qquad \text{or} \qquad c_1 = 0 \tag{E12}$$

Substituting Equations (E11) and (E12) into Equation (E10) and integrating twice, we obtain

$$EI_{zz}\frac{dv}{dx} = \frac{5}{6}\langle x\rangle^3 - \frac{5}{6}\langle x-2\rangle^3 - \frac{2}{3}\langle x-3\rangle^3 - \frac{5}{2}\langle x-2\rangle^2 - 12\langle x-2\rangle^1 + c_3 \tag{E13}$$

$$EI_{zz}v = \frac{5}{24}\langle x\rangle^4 - \frac{5}{24}\langle x-2\rangle^4 - \frac{2}{12}\langle x-3\rangle^4 - \frac{5}{6}\langle x-2\rangle^3 - 6\langle x-2\rangle^2 + c_3 x + c_4 \tag{E14}$$

Substituting Equation (E14) into Equation (E5), we obtain

$$c_4 = 0 \tag{E15}$$

Substituting Equation (E14) into Equation (E6), we obtain

$$\frac{5}{24}\langle 6\rangle^4 - \frac{5}{24}\langle 4\rangle^4 - \frac{2}{12}\langle 3\rangle^4 - \frac{5}{6}\langle 4\rangle^3 - 6\langle 4\rangle^2 + c_3(6) = 0 \qquad \text{or} \qquad c_3 = -\frac{323}{36} = -8.97 \tag{E16}$$

Substituting Equations (E15) and (E16) into Equation (E14) and simplifying, we obtain the elastic curve,

$$v = \frac{1}{72EI_{zz}}[15\langle x\rangle^4 - 15\langle x-2\rangle^4 - 12\langle x-3\rangle^4 - 60\langle x-2\rangle^3 - 432\langle x-2\rangle^2 - 646x] \tag{E17}$$

Substituting $x = 2$ into Equation (E17), we obtain the deflection at point *B*,

$$v(2) = \frac{1}{72[(42)(10^3)]}[15\langle 2\rangle^4 - 15\langle 0\rangle^4 - 12\langle -1\rangle^4 - 60\langle 0\rangle^3 - 432\langle 0\rangle^2 - 646(6)] \tag{E18}$$

ANS. $v(2) = -1.2$ mm

(b) As stated in Equation (7.1), the moment M_z can be found from Equation (E10). And as seen in Equation (7.4), the shear force V_y is the negative of the expression given in Equation (E11). Noting that the constants c_1 and c_2 are zero, we obtain the expressions for M_z and V_y:

$$M_z(x) = \left[\frac{5}{2}\langle x\rangle^2 - \frac{5}{2}\langle x-2\rangle^2 - 2\langle x-3\rangle^2 - 5\langle x-2\rangle^1 - 12\langle x-2\rangle^0\right] \text{ kN}\cdot\text{m} \tag{E19}$$

$$V_y(x) = [-5\langle x\rangle^1 + 5\langle x-2\rangle^1 + 4\langle x-3\rangle^1 + 5\langle x-2\rangle^0 + 12\langle x-2\rangle^{-1}] \text{ kN} \tag{E20}$$

Point B is at $x = 2$. Just after point B, that is, at $x = 2-$, all terms except the first term in Equations (E19) and (E20) are zero.

ANS. $M_z(2-) = 10 \text{ kN}\cdot\text{m}$ $V_y(2-) = -10 \text{ kN}$

Just after point B, that is, at $x = 2+$, the step function $\langle x-2\rangle^0$ is equal to 1. Hence this term along with the first term are the nonzero terms in Equations (E19) and (E20).

$$M_z(2+) = (10-12) = -2 \text{ kN}\cdot\text{m} \qquad V_y(2+) = (-10+5) = -5 \text{ kN} \tag{E21}$$

ANS. $M_z(2+) = -2 \text{ kN}\cdot\text{m}$ $V_y(2+) = -5 \text{ kN}$

COMMENT

1. We note that $M_z(2+) - M_z(2-) = -12 \text{ kN}\cdot\text{m}$ and $V_y(2+) - V_y(2-) = 5 \text{ kN}\cdot\text{m}$, which are the values of the applied moment and applied shear force. Thus the jump in the internal shear force and internal moment difference is captured by the step function.

7.5* AREA-MOMENT METHOD

One last method is especially useful in finding the deflection or the slope of the beam is to be found at a specific point. Called the *area-moment method*, it is based on graphical interpretation of the integrals that are generated by integration of Equation (7.1).

Equation (7.1) can be written

$$\frac{d}{dx}v'(x) = \frac{M_z}{EI_{zz}} \tag{7.20.a}$$

where $v'(x) = dv(x)/dx$ represents the slope of the elastic curve. Integrating the equation from any point A to any other point x, we obtain

$$\int_{v'(x_A)}^{v'(x)} dv'(x) = \int_{x_A}^{x} \frac{M_z}{EI_{zz}}\,dx_1 \quad \text{or}$$

$$v'(x) = v'(x_A) + \int_{x_A}^{x} \frac{M_z}{EI_{zz}}\,dx_1 \tag{7.20.b}$$

Integrating Equation (7.20.b) between point A and any point x, we obtain

$$v(x) = v(x_A) + v'(x_A)(x-x_A) + \int_{x_A}^{x}\left(\int_{x_A}^{x} \frac{M_z}{EI_{zz}}\,dx_1\right)dx \tag{7.20.c}$$

The last integral[2] can be written as

$$v(x) = v(x_A) + v'(x_A)(x-x_A) + \int_{x_A}^{x}(x-x_1)\frac{M_z}{EI_{zz}}\,dx_1 \tag{7.20.d}$$

Assume EI_{zz} is a constant for the beam. From Equations 7.20.b and 7.20.d, the slope and the deflection at point B can be written

[2]By integrating by parts, it can be shown that

$$\int_{x_A}^{x}\left[\int_{x_A}^{x} f(x_1)dx_1\right]dx = \int_{x_A}^{x}(x-x_1)f(x_1)\,dx_1$$

Letting $f(x_1) = M_z/EI_{zz}$, we can obtain Equation (7.20.d) from Equation (7.20.c).

$$v'(x_B) = v'(x_A) + \frac{1}{EI_{zz}}\int_{x_A}^{x_B} M_z\, dx \tag{7.21}$$

$$v(x_B) = v(x_A) + v'(x_A)(x_B - x_A) + \frac{1}{EI_{zz}}\int_{x_A}^{x_B} (x_B - x)\, M_z\, dx \tag{7.22}$$

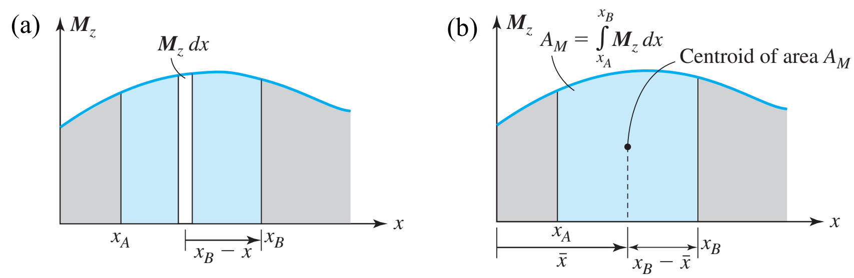


Figure 7.43 Graphical interpretation of integrals in area-moment method.

The integral in Equation (7.21) can be interpreted as the area under the bending moment curve, as shown in Figure 7.43. The moment diagram can be constructed as discussed in Section 6.4. Thus, if the slope $v'(x_A)$ at point *A* is known, then by adding the area under the moment curve, we can obtain the slope at point *B*. The area A_M will be considered positive if the moment curve is in the upper plane and negative if it is in the lower plane.

From Figure 7.43*a*, we see that the integral in Equation (7.22) is the first moment of the area under the moment curve about point *B*. This first moment of the area can be found by taking the distance of the centroid from *B* and multiplying by the area,

$$\int_{x_A}^{x_B} (x_B - x) M_z\, dx = (x_B - \bar{x}) A_M \tag{7.23.a}$$

With this interpretation the deflection of *B* can be found from Equation (7.22). Table C.2 in the Appendix lists the areas and the centroids of the areas under various curves. These values can be used in calculating the integrals in Equations 7.21 and 7.22.

Consider the cantilever beam in Figure 7.44*a* and the associated bending moment diagram. At point *A* the slope and the deflection at *A* are zero. Hence $v'(x_A) = 0$ and $v(x_A) = 0$ in Equations 7.21 and 7.22. The area A_M, representing the integral in Equation (7.21), is $-PL(L)/2$. Thus the slope at *B* is $v'(x_B) = -PL^2/2EI$. Since distance of the centroid from *B* is $x_B - \bar{x} = 2L/3$, $(x_B - \bar{x})A_M = (2L/3)(-PL^2/2)$ is the value of the integral equation, Equation (7.22). Thus the deflection at *B* is $v(x_B) = -PL^3/3EI$.

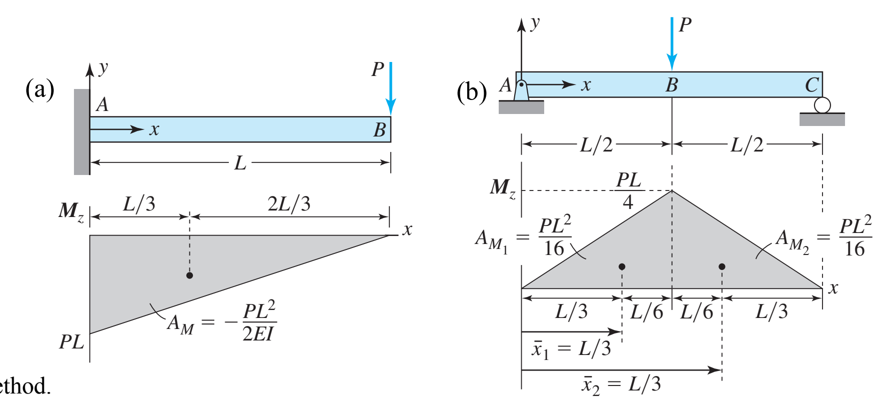


Figure 7.44 Application of area-moment method.

Now consider the simply supported beam and the associated bending moment in Figure 7.44*b*. The value of the slope is not known at any point on the beam. Thus before the deflection and slope at *B* can be determined, the slope at *A* must be found. The deflection at *A* is zero. Treating the slope at *A* as an unknown constant, we equate the deflection at *C* from Equation (7.22) to zero and obtain the slope at *A*,

$$v(x_C) = v(x_A) + v'(x_A)(x_C - x_A) + \frac{1}{EI}[A_{M_1}(x_C - \bar{x}_1) + A_{M_2}(x_C - \bar{x}_2)] = 0 \text{ or} \tag{7.23.b}$$

$$v'(x_A)(L) + \frac{1}{EI}\left[\frac{PL^2}{16}\left(\frac{2L}{3}\right) + \frac{PL^2}{16}\left(\frac{L}{3}\right)\right] = 0 \qquad \text{or} \qquad v'(x_A) = -\frac{PL^2}{16EI} \tag{7.23.c}$$

Using Equation (7.22) once more, we find the deflection at point B:

$$v(x_B) = v(x_A) + v'(x_A)(x_B - x_A) + \frac{1}{EI}[A_{M_1}(x_B - \bar{x}_1)]$$

$$= -\frac{PL^2}{16EI}\left(\frac{L}{2}\right) + \frac{1}{EI}\left[\frac{PL^2}{16}\left(\frac{L}{6}\right)\right] = -\frac{PL^3}{48} \tag{7.23.d}$$

EXAMPLE 7.13

In terms of E, I, w, and L, determine the deflection and slope at point B for the beam and loading shown in Figure 7.45.

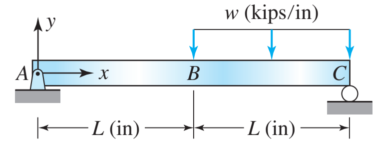


Figure 7.45 Beam and loading in Example 7.13.

PLAN

The reaction forces at A and C can be found and then the shear–moment diagram can be drawn as discussed in Section 6.4. The area under the moment curve and the location of the centroids can then be determined. Because the deflection at A is zero, the deflection at C can be written in terms of the unknown slope at A using Equation (7.22). Equating the deflection at C to zero, then gives the slope at A. Slope and deflection at B can now be found using Equations 7.21 and 7.22, respectively.

SOLUTION

From the free-body diagram of the entire beam, the reaction forces at the supports can be found and the shear–moment diagram drawn, as shown in Figure 7.46. The moment curve in region BC is a quadratic, and the areas under the curves is the sum of three area. Table C.2 in the Appendix lists the formulas for the areas and centroids.

$$A_1 = \frac{L}{2}\left(\frac{wL^2}{4}\right) = \frac{wL^3}{8} \qquad \bar{x}_1 = \frac{2}{3}(L) \tag{E1}$$

$$A_2 = \frac{L}{4}\left(\frac{wL^2}{4}\right) = \frac{wL^3}{16} \qquad \bar{x}_2 = L + \frac{L}{8} = \frac{9L}{8} \tag{E2}$$

$$A_3 = \frac{2}{3}\left(\frac{L}{4}\right)\left(\frac{wL^2}{32}\right) = \frac{wL^3}{192} \qquad \bar{x}_3 = L + \frac{5}{8}\left(\frac{L}{4}\right) = \frac{37L}{32} \tag{E3}$$

$$A_4 = \frac{2}{3}\left(\frac{3L}{4}\right)\left(\frac{9wL^2}{32}\right) = \frac{9wL^3}{64} \qquad \bar{x}_4 = 2L - \frac{5}{8}\left(\frac{3L}{4}\right) = \frac{49L}{32} \tag{E4}$$

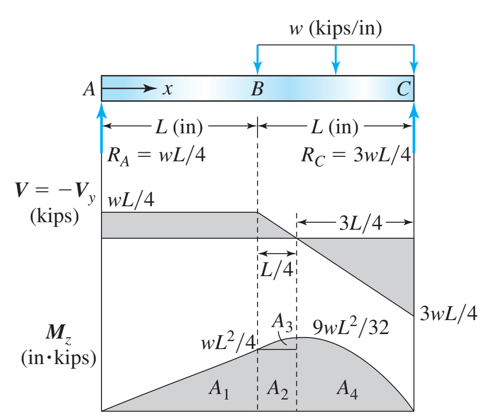


Figure 7.46 Shear–moment diagram in Example 7.13.

The deflection at C can be written as

$$v(x_C) = v(x_A) + v'(x_A)(x_C - x_A) + \frac{1}{EI}[A_1(x_C - \bar{x}_1) + A_2(x_C - \bar{x}_2) + A_3(x_C - \bar{x}_3) + A_4(x_C - \bar{x}_4)] \quad \text{(E5)}$$

The deflections at the support are zero. Substituting $v(x_C) = 0$ and $v(x_A) = 0$ and the values of the areas and centroids in Equation (E5), we can find the slope at A.

$$v'(x_A)(2L) + \frac{1}{EI}\left[\frac{wL^3}{8}\left(2L - \frac{2L}{3}\right) + \frac{wL^3}{16}\left(2L - \frac{9L}{8}\right) + \frac{wL^3}{192}\left(2L - \frac{37L}{32}\right) + \frac{9wL^3}{64}\left(2L - \frac{49L}{32}\right)\right] = 0 \quad \text{or} \quad v'(x_A) = -\frac{7wL^3}{48EI} \quad \text{(E6)}$$

The deflection at B can be written as

$$v(x_B) = v(x_A) + v'(x_A)(x_B - x_A) + \frac{1}{EI}[A_1(x_B - \bar{x}_1)] \quad \text{(E7)}$$

Substituting the calculated values we obtain the deflection at B,

$$v(x_B) = -\frac{7wL^3}{48EI}(L) + \frac{1}{EI}\left(\frac{wL^3}{8}\right)\left(L - \frac{2L}{3}\right) \quad \text{(E8)}$$

ANS. $v(x_B) = -\frac{5wL^4}{48EI}$

COMMENTS

1. The example demonstrates uses of the area moment method for finding slopes and deflection at a point in the beam.
2. If the elastic curve needs to be determined for an indeterminate beam, we can use the area moment method to determine the reactions and then the second-order differential equation to solve the problem. But if this approach is to have any computational advantage over using the fourth-order differential equations, then it must be possible to draw the moment diagram quickly by inspection.

PROBLEM SET 7.3

Superposition

7.57 Determine the deflection at the free end of the beam shown in Figure P7.20.

7.58 Determine the reaction force at support A in Figure P7.34.

7.59 Determine the deflection at point A on the beam shown in Figure P7.59 in terms of w, L, E, and I.

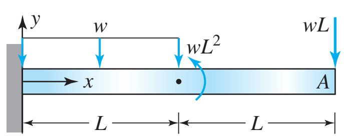


Figure P7.59

7.60 Determine the reaction force and the slope at A for the beam shown in Figure P7.60, using superposition.

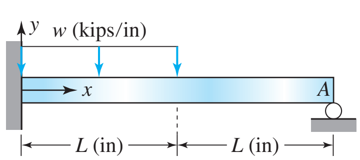


Figure P7.60

7.61 Two beams of length L and bending rigidity EI, shown in Figure P7.61, are simply supported at the ends and are in contact at the center. Determine the deflection at the center in terms of P, L, E, and I.

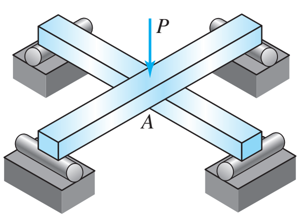


Figure P7.61

7.62 Two beams of length L and bending rigidity EI, shown in Figure P7.62, are simply supported at the ends and are in contact at the center. Determine the deflection at the center in terms of w, L, E, and I.

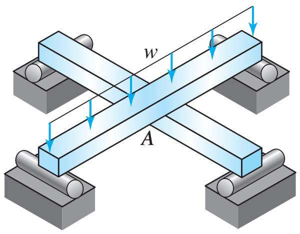


Figure P7.62

7.63 A cantilever beam's end rests on the middle of a simply supported beam, as shown in Figure P7.63. Both beams have length L and bending rigidity EI. Determine the deflection at A and the reactions at the wall at C in terms of P, L, E, and I.

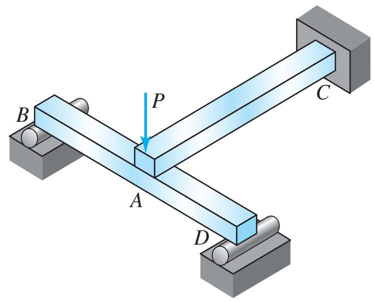


Figure P7.63

7.64 A cantilever beam's end rests on the middle of a simply supported beam, as shown in Figure P7.64. Both beams have length L and bending rigidity EI. Determine the deflection at A and the reactions at the wall at C in terms of w, L, E, and I.

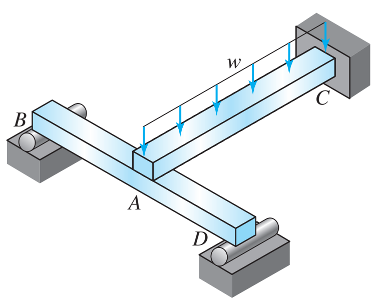


Figure P7.64

7.65 The end of one cantilever beam rests on the end of another cantilever beam, as shown in Figure P7.65. Both beams have length L and bending rigidity EI. Determine the deflection at A and the reactions at the wall at C in terms of w, L, E, and I.

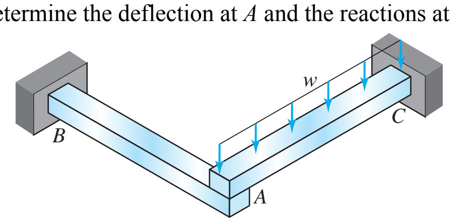


Figure P7.65

Discontinuity functions

7.66 A gymnast with a mass of 60 kg stands in the middle on a wooden balance beam as shown in Figure P7.66. The modulus of elasticity of the wood is 12.6 GPa. To bracket the elasticity of the support, two models are to be considered: (*a*) the supports are simply supported; (*b*) the supports are built in ends. Determine the maximum deflection of the beam for both the cases.

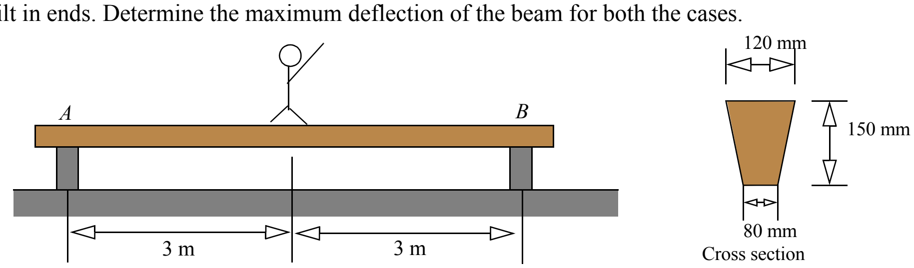


Figure P7.66

7.67 Solve Problem 7.17 using discontinuity functions.

7.68 Solve Problem 7.18 using discontinuity functions.

7.69 Solve Problem 7.19 using discontinuity functions.

7.70 Solve Problem 7.20 using discontinuity functions.

7.71 (a) Solve for the elastic curve for the beam and loading shown in Figure P7.23. (b) Determine the slope and deflection at point *C*.

7.72 Solve Problem 7.34 using discontinuity functions.

7.73 A beam is supported and loaded as shown in Figure P7.73. The spring constant in terms of beam stiffness is written as $k = \alpha EI/L^3$, where α is a proportionality factor. Determine the extension of the spring in terms of α, w, E, I, and L.

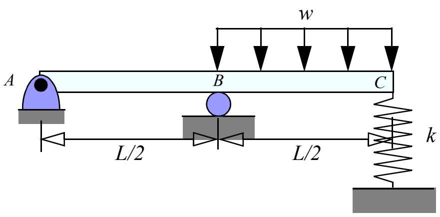


Figure P7.73

Area-moment method

7.74 Using the area-moment method, determine the deflection in the middle for the beam shown in Figure P7.2.

7.75 Using the area-moment method, determine the deflection in the middle of the beam shown in Figure P7.3.

7.76 Using the area-moment method, determine the deflection and slope at the free end of the beam shown in Figure P7.4.

7.77 Using the area-moment method, determine the slope at $x = 0$ and deflection at $x = L$ of the beam shown in Figure P7.6.

7.78 Using the area-moment method, determine the slope at $x = 0$ and deflection at $x = L$ of the beam shown in Figure P7.17.

7.79 Using the area-moment method, determine slope at $x = 0$ and deflection at $x = L$ of the beam shown in Figure P7.18.

7.80 Using the area-moment method, determine the slope at the free end of the beam shown in Figure P7.20.

Stretch Yourself

7.81 To improve the load carrying capacity of a wooden beam (E_W = 2000 ksi) a steel strip (E_s = 30,000 ksi) is securely fastened to it as shown in Figure P7.81. Determine the deflection at $x = L$.

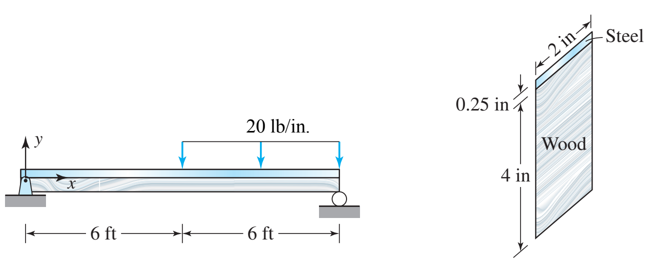


Figure P7.81

*7.6 CONCEPT CONNECTOR

Compared to a theory for the deflection of beams, our understanding of the strength of beams developed more intuitively, as described in Section 6.7. The very term *elastic curve* for the deflection of a beam reflects the early impact of mathematics.

7.6.1 History: Beam Deflection

In the seventeenth century, procedures were in use to draw tangents (similar to differentiation) and areas swept by curves (similar to integration). Issac Newton (1642-1727) in 1666 realized that the two procedures were inverse of each other and developed a method called *fluxional method*, which he circulated among some of his friends but did not publish. Newton's method did not receive much attention. Nine years later in Germany, Gottfried Wilhelm Leibniz (1646-1716) developed nearly the same method independently. Leibniz's notation caught on, especially in Europe, and so did his name for the method, *differential calculus*. Members of the Bernoulli family were in the forefront of finding applications for this new mathematical tool. One of the applications they considered was the determination of the elastic curve.

Daniel Bernoulli

Leonard Euler

Figure 7.47 Pioneers of beam deflection theories.

Jacob Bernoulli (1646–1716) and John Bernoulli (1667–1748) won acclaim for their mathematical work, which the French Academy of Science recognized by making the brothers members in 1699. Daniel Bernoulli (1700–1782), John's son, made important contributions to hydrodynamics, while Leonard Euler (1707–1782), John's pupil, introduced analytic methods used today in practically every area of mathematics. His name is also associated with buckling theory, as we shall see in Chapter 11. Both Daniel Bernoulli and Euler were pioneers in the theory of the elastic curve.

Jacob Bernoulli had started with Mariotte's assumption that the neutral axis is tangent to the bottom (the concave side) of the curve in a cantilever beam. From this he obtained a relationship between the curvature of the beam at any point and the applied load, as described in Problem 7.46. Although Mariotte's assumption proved incorrect, Bernoulli's result was correct—except for the value of the bending rigidity. Euler, on the suggestion of Daniel Bernoulli, approached the same problem by minimizing the strain energy in a beam, which yielded the correct relationship. Euler called the constant relating moment and curvature the *moment of stiffness* (rather than bending rigidity), but he recognized that it had to be determined experimentally. As we saw in Section 3.12.1, much later Thomas Young made a similar observation concerning axial rigidity, and the modulus of elasticity is named after him. Such are the quirks of history.

Claude-Louis Navier (1785–1836), whose work on the concept of stress we met in Section 1.5, was the first to solve for the deflection of statically indeterminate beams. Navier carried the extra unknown reactions in the second-order differential equation and determined these reactions from conditions on the deflection and slopes at the support (see Problem 7.45).

Jean Claude Saint-Venant, whose work we have seen in several chapters, analyzed the deflection of a cantilever beam due to a force at the free end. He was the first to realize that it can be found without formally integrating the differential equations. This was the beginning of the area-moment method that we studied in Section 7.5*. Alfred Clebsch (1833–1872), in his 1862 book on elasticity, considered the deflection of a beam under concentrated forces (see Problem 7.47). His approach later evolved into the discontinuity method, discussed in Section 7.4*. The English mathematician W. H. Macaulay formally introduced the discontinuity functions in 1919.

Each aspect of beam theory thus had its own development. The normal stress in bending was relatively intuitive; shear stress in bending was guided by experiment; and beam deflection was guided by mathematics. Together, they highlight the importance of intuition, experimental evidence, and mathematical formalization. Engineers need them all to understand nature.

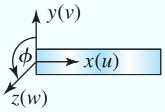
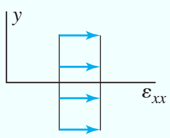
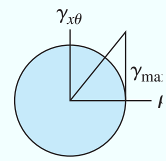
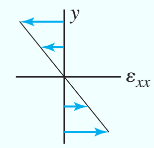
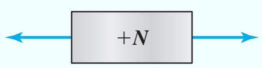
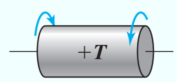
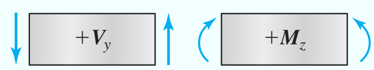
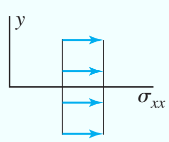
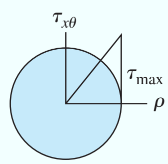
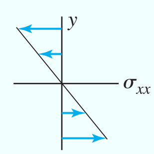
7.7 CHAPTER CONNECTOR

In this chapter we saw several methods for determining the deflection of beams. The preferred approach treats beam deflection as a second-order boundary-value problem. However, other approaches may be needed if the distributed loads on the beam are complicated functions or if we have only experimentally measured values for the distributed load. With the discontinuity function, a single differential equation can represent the loading on the entire beam. This method should be used if the beam loading changes in a discrete manner across the beam. The area-moment method, a graphical technique, can yield quick solutions for the beam deflection and slope at a point *if* the moment diagram can be constructed easily. The superposition method is another versatile design tool. It can be used for determinate or indeterminate beams, provided we know the beam deflection and slope. Handbooks supply these values for many basic cases.

Chapter 7 concludes the second major part of this book. Table 7.1 offers a synopsis of one-dimensional structural elements, described in Chapters 4 through 7. The table highlights the essential elements common to these theories. They allow us to obtain deformation, strains, and stresses at any point in a one-dimensional structural element. In the next three chapters we will use this information in many ways.

In order to determine whether a structure will break under a given load, we need *failure theories*, which we study in Chapter 10. To apply failure theories, we first need to determine the maximum normal and shear stresses at a point. Chapter 8, on *stress transformation*, describes how to obtain these stresses from our one-dimensional theories. Only experiment, however, can render the final verdict on designs based on the one-dimensional theory. One popular experimental technique is to measure strains using *strain gages*. And this technique requires a relationship between the strains obtained from one-dimensional theory and the strains in any given direction. That relationship, known as *strain transformation*, is the topic of Chapter 9. Chapter 10 is the culmination of the first nine chapters. Here we study stresses and strains in structural elements subject to combined axial, torsional, and bending loads. We also address the design and failure of structures and machine elements.

Table 7.1 Synopsis of one-dimensional structural theories.

	Axial (Rods)	Torsion (Shafts)	Symmetric Bending (Beams)
Displacements/ deformation	$u(x, y, z) = u(x)$ $v = 0 \quad w = 0$	$u = 0 \quad v = 0 \quad w = 0$ $\phi(x, y, z) = \phi(x)$	$u(x, y, z) = -y\dfrac{dv}{dx}$ $v = v(x) \quad w = 0$
Strains	$\varepsilon_{xx} = \dfrac{du}{dx}$ 	$\gamma_{x\theta} = \rho\dfrac{d\phi}{dx}$ 	$\varepsilon_{xx} = -y\dfrac{d^2v}{dx^2}$ 
Stresses	$\sigma_{xx} = E\varepsilon_{xx} = E\dfrac{du}{dx}$	$\tau_{x\theta} = G\gamma_{x\theta} = \rho\dfrac{d\phi}{dx}$	$\sigma_{xx} = E\varepsilon_{xx} = -Ey\dfrac{d^2v}{dx^2} \qquad \tau_{xy} \neq 0$
Internal forces and moments	$\boldsymbol{N} = \int_A \sigma_{xx}\, dA$	$\boldsymbol{T} = \int_A \rho\tau_{x\theta}\, dA$	$\int_A \sigma_{xx}\, dA = 0$ ...Locates neutral axis $\boldsymbol{M}_z = -\int_A y\sigma_{xx}\, dA \qquad \boldsymbol{V}_y = \int_A \tau_{xy}\, dA$
Sign convention			
Stress formulas	$\sigma_{xx} = \dfrac{\boldsymbol{N}}{A}$ 	$\tau_{x\theta} = \dfrac{\boldsymbol{T}\rho}{J}$ 	$\sigma_{xx} = -\dfrac{\boldsymbol{M}_z y}{I_{zz}}$  $\tau_{sx} = \tau_{xs} = -\dfrac{\boldsymbol{V}_y Q_z}{I_{zz}t}$
Deformation formulas	$\dfrac{du}{dx} = \dfrac{N}{EA}$ $u_2 - u_1 = \dfrac{\boldsymbol{N}(x_2 - x_1)}{EA}$ EA = axial rigidity	$\dfrac{d\phi}{dx} = \dfrac{\boldsymbol{T}}{GJ}$ $\phi_2 - \phi_1 = \dfrac{\boldsymbol{T}(x_2 - x_1)}{GJ}$ GJ = torsional rigidity	$\dfrac{d^2v}{dx^2} = \dfrac{\boldsymbol{M}_z}{EI_{zz}}$ $v = \int\left(\int \dfrac{\boldsymbol{M}_z}{EI_{zz}}\, dx\right) dx + C_1x + C_2$ EI_{zz} = bending rigidity

POINTS AND FORMULAS TO REMEMBER

- The deflected curve of a beam represented by v(x) is called *elastic curve.*
- $\boldsymbol{M}_z = EI_{zz}\dfrac{d^2\mathrm{v}}{dx^2}$ (7.1) $\quad \boldsymbol{V}_y = -\dfrac{\mathrm{d}}{\mathrm{d}x}\left(EI_{zz}\dfrac{d^2\mathrm{v}}{dx^2}\right)$ (7.4) $\quad \dfrac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI_{zz}\dfrac{d^2\mathrm{v}}{dx^2}\right) = p$ (7.5)
- where v is the deflection of the beam at any x and is positive in the positive y direction; $\boldsymbol{M}_z$ is the internal bending moment; $\boldsymbol{V}_y$ is the internal shear force; p is the distributed force on the beam and is positive in the positive y direction; EI_{zz} is the bending rigidity of the beam; and $d^2\mathrm{v}/dx^2$ is the curvature of the beam.
- The mathematical statement listing all the differential equations and all the conditions necessary for solving for v(x) is called *boundary-value problem* for beam deflection.
- Boundary conditions for second-order differential equations:

  Built-in end at x_A $\mathrm{v}(x_A) = 0 \qquad \dfrac{\mathrm{dv}}{\mathrm{d}x}(x_A) = 0$

  Simple support at x_A $\mathrm{v}(x_A) = 0$

  Smooth slot at x_A $\dfrac{\mathrm{dv}}{\mathrm{d}x}(x_A) = 0$
- Continuity conditions at x_j:

  $$\mathrm{v}_1(x_j) = \mathrm{v}_2(x_j) \qquad \frac{d\mathrm{v}_1}{dx}(x_j) = \frac{d\mathrm{v}_2}{dx}(x_j)$$
- Boundary conditions for fourth-order differential equations are determined at each boundary point by specifying: $(\mathrm{v} \text{ or } \boldsymbol{V}_y)$ and $(d\mathrm{v}/dx \text{ or } \boldsymbol{M}_z)$ (7.6)
- In fourth-order boundary-value problems, at each point x_j where the differential equation changes, the continuity conditions and equilibrium conditions must be specified.
- The superposition method is a versatile design tool that can be used for solving problems of determinate and indeterminate beams provided the beam deflection and slope values are available for many basic cases, such as in a handbook.
- In the discontinuity function method a single differential equation and conditions on deflection and slopes at support describe the complete boundary-value problem.
- Discontinuity functions:

  $$\langle x-a\rangle^{-n} = \begin{cases} 0 & x \neq a \\ \infty & x \to a \end{cases} \qquad \langle x-a\rangle^{n} = \begin{cases} 0 & x \le a \\ (x-a)^n & x > a \end{cases}$$
- Differentiation formulas:

  $$\frac{d\langle x-a\rangle^{-1}}{dx} = \langle x-a\rangle^{-2} \qquad \frac{d\langle x-a\rangle^{0}}{dx} = \langle x-a\rangle^{-1} \qquad \frac{d\langle x-a\rangle^{n}}{dx} = n\langle x-a\rangle^{n-1} \quad n \ge 1$$
- Integration formulas:

  $$\int_{-\infty}^{x}\langle x-a\rangle^{-2}\,dx = \langle x-a\rangle^{-1} \qquad \int_{-\infty}^{x}\langle x-a\rangle^{-1}\,dx = \langle x-a\rangle^{0} \qquad \int_{-\infty}^{x}\langle x-a\rangle^{n}dx = \frac{\langle x-a\rangle^{n+1}}{n+1} \quad n \ge 0$$
- The area-moment method is a graphical technique that can yield quick solutions of beam deflection and slope at a point, *if* the moment diagram can be constructed easily.

  $$\mathrm{v}'(x_B) = \mathrm{v}'(x_A) + \frac{1}{EI_{zz}}\underbrace{\int_{x_A}^{x_B}\boldsymbol{M}_z\,dx}_{A_M} \qquad (7.21)$$

  $$\mathrm{v}(x_B) = \mathrm{v}(x_A) + \mathrm{v}'(x_A)(x_B - x_A) + \frac{1}{EI_{zz}}\underbrace{\int_{x_A}^{x_B}(x_B - x)\boldsymbol{M}_z\,dx}_{(x_B-\bar{x})A_M} \qquad (7.22)$$