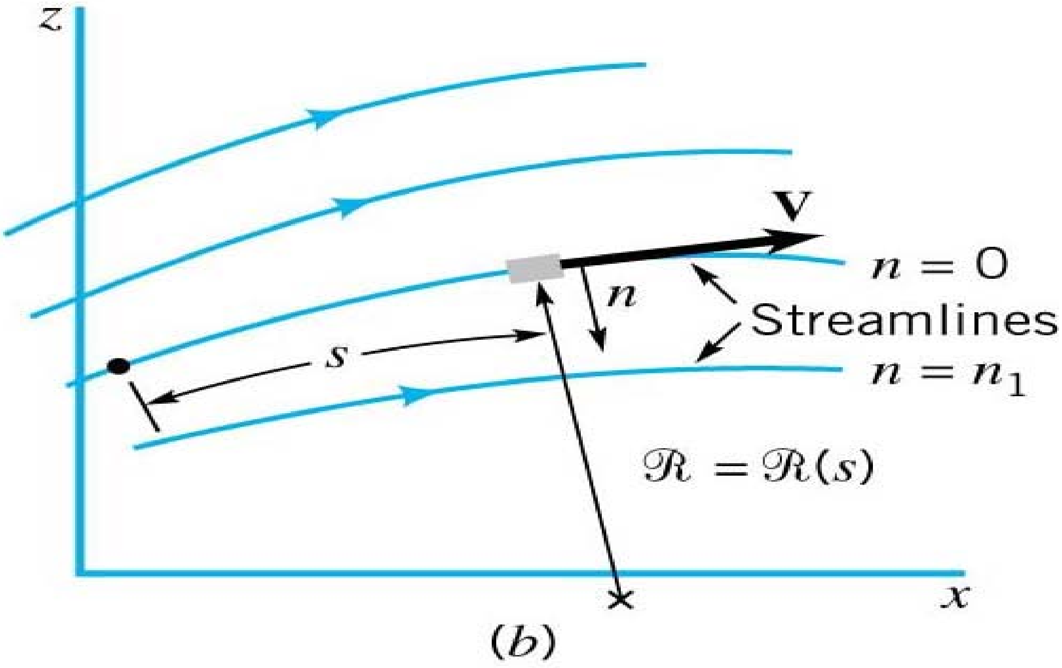
Ch3	The Bernoulli Equation	

The most used and the most abused equation in fluid mechanics.

3.1 Newton's Second Law: $\vec{F} = ma$

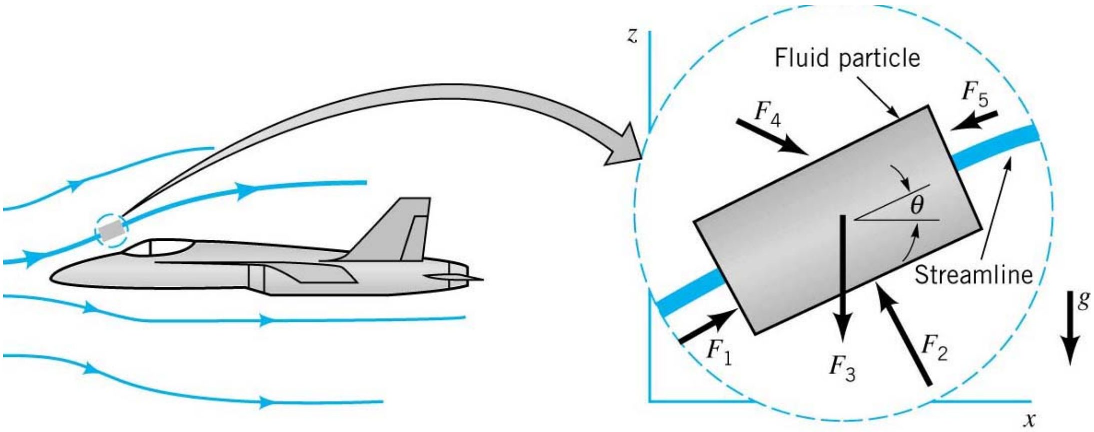
- In general, most real flows are 3-D, unsteady (x, y, z, t; r, θ, z, t; etc)
- Let consider a 2-D motion of flow along "streamlines", as shown below.



- Velocity ($\vec{V}$): Time rate of change of the position of the particle.
- Streamlines: The lines that are tangent to the velocity vectors throughout the flow field.
- Note: For steady flows, each particle slide along its path, and its velocity vector is everywhere tangent to the path.
- Streamline coordinate: $S = S(t)$; $\vec{V} = dS/dt$ (the distance along the streamline can be decided by $\vec{V}$ and $R(s)$)

• By chain rule, $\underline{\underline{a_s = \frac{dV}{dt} = \frac{\partial V}{\partial S}\frac{dS}{dt} = V\frac{\partial V}{\partial S}}}$; $\underline{\underline{a_n = \frac{V^2}{R}}}$

where R is the local radius of curvature of the streamline, and S is the distance measured along the streamline from some arbitrary initial point.



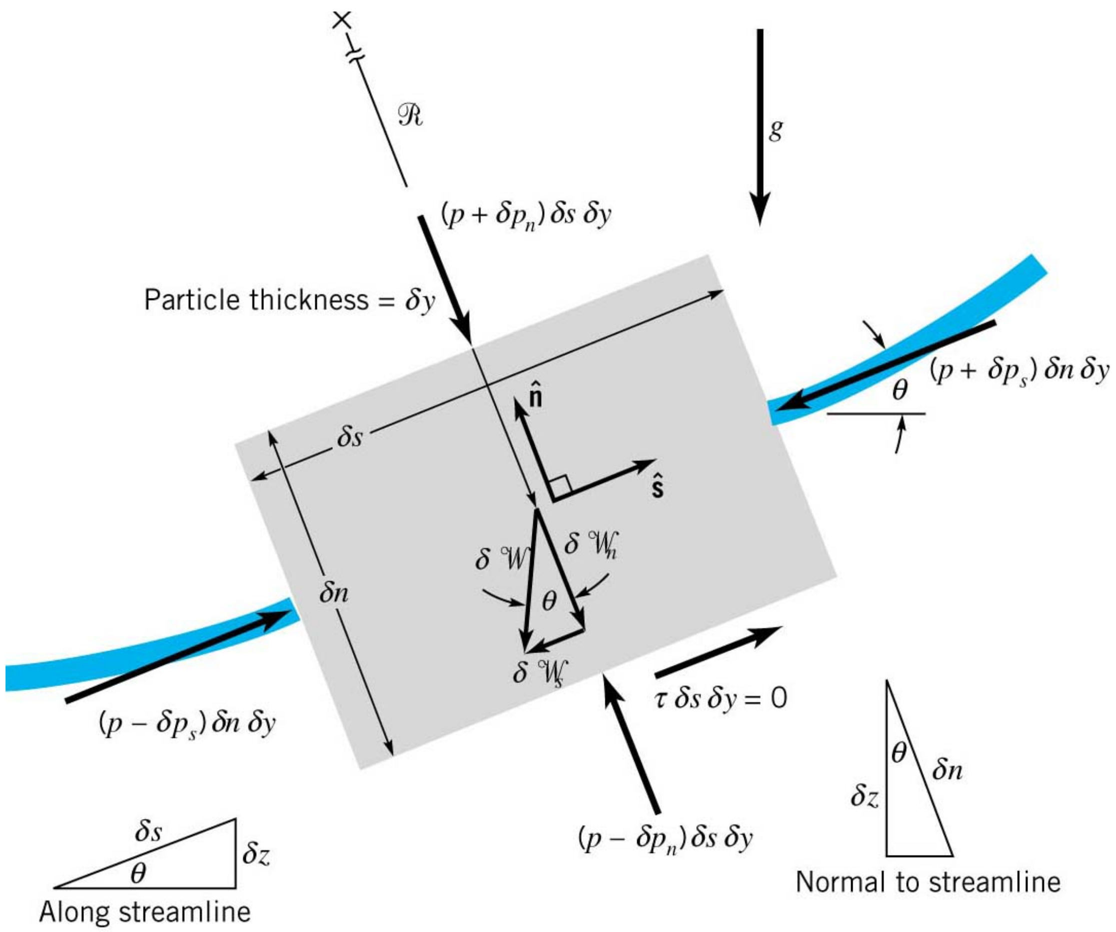
■ Figure 3.2 Isolation of small fluid particle in a flow field.

$\frac{\partial V}{\partial S} \neq 0; \quad R \neq \infty$ (not straight line)

Forces: Gravity & Pressure (important)

Viscous, Surface tension (negligible)

3.2 $\vec{F} = m\vec{a}$ along a streamline



■ **Figure 3.3 Free-body diagram of a fluid particle for which the important forces are those due to pressure and gravity.**

· Consider the small fluid particle ($\delta S \times \delta n$), as show above.

If the flow is steady.

$$\text{N.} \quad \Rightarrow \sum \delta F_S = \delta m \cdot a_S = \delta m V \frac{\partial V}{\partial S} = \rho \delta \forall V \frac{\partial V}{\partial S} \qquad (3.2)$$

(Eq. (3.2) is valid for both compressible and incompressible fluids)

$$\text{W.} \Rightarrow \delta W_S = -\delta W \sin\theta = -\underline{\gamma \delta \sin\theta} \qquad (\theta = 0 \Rightarrow \delta \mathrm{W_S} = 0)$$

P. $\Rightarrow \delta P_S \approx \dfrac{\partial P}{\partial S}\dfrac{\delta S}{2}$ (first term of Taylor series expansion, because the particle is small; $P+\delta P_S \approx P-\delta P_S$)

Thus δF_{PS} : the net pressure force on the particle in the streamline direction

$$\begin{aligned}\delta F_{PS} &= (P-\delta P_S)\delta n\,\delta y-(P+\delta P_S)\delta n\,\delta y\\ &= -2\delta P_S\delta n\,\delta y = -\frac{\partial P}{\partial S}\delta s\,\delta n\,\delta y\\ &= -\frac{\partial P}{\partial S}\delta\forall\end{aligned}$$

*Note: if pressure gradient is not zero ($P \neq C$), then there is a net pressure force. $\nabla P = \dfrac{\partial P}{\partial S}\vec{S} + \dfrac{\partial P}{\partial n}\vec{n}$

Net Force $\Rightarrow \sum \delta F_S = \delta W_S + \delta F_{PS}$;

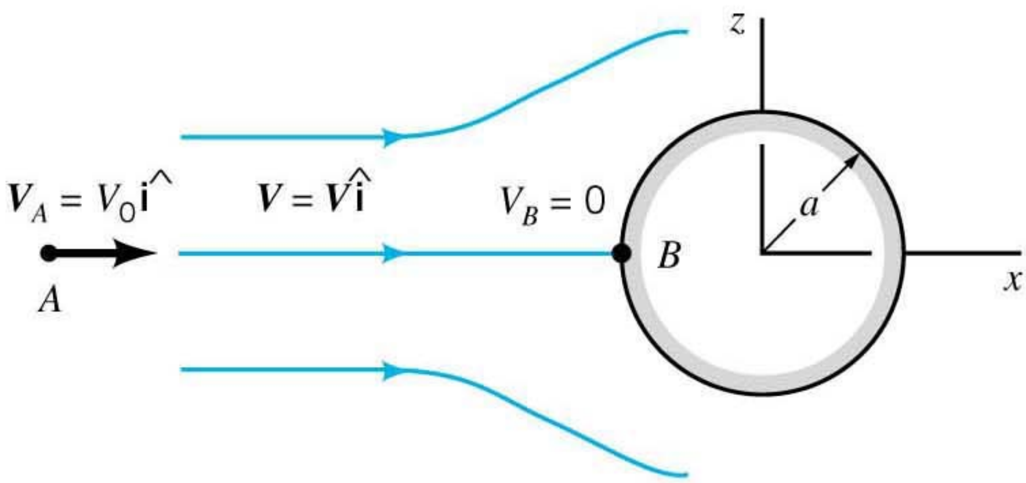
$$\underline{\rho V\frac{\partial V}{\partial S} = -\gamma \sin\theta - \frac{\partial P}{\partial S}} \qquad (3.4)$$

Note: Force balancing consideration ρV (mass flux), not ρ or V, is a key parameter for fluid mechanics, if $\rho \neq$ constant.

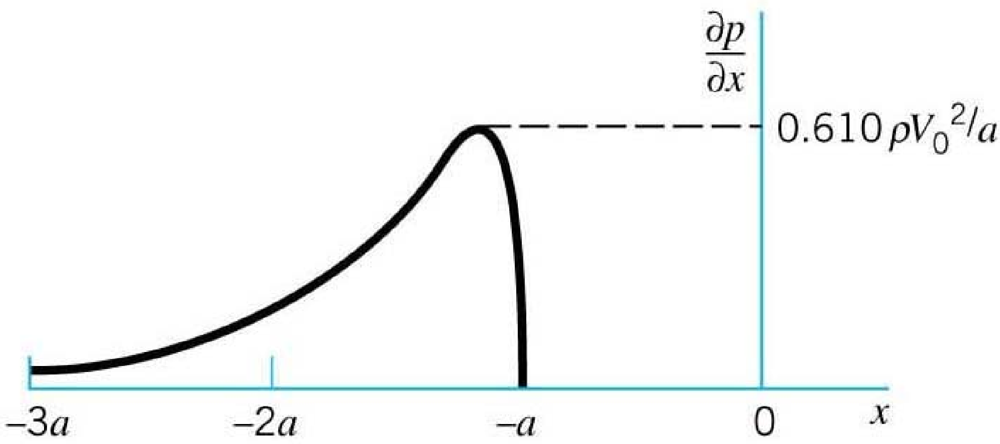
Example 3.1 Consider the inviscid, incompressible, steady flow along the horizontal streamline A-B in front of the sphere of radius a as shown in Fig. E3.1a. From a more advanced theory of flow past a sphere, the fluid velocity along this streamline is

$$V = V_0 \left(1 + \frac{a^3}{x^3}\right)$$

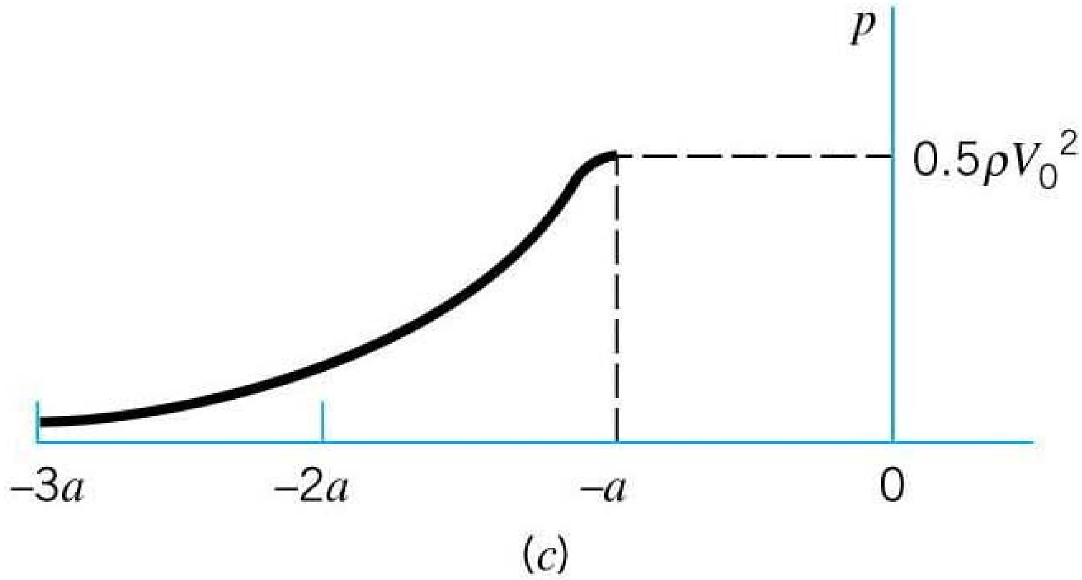
Determinate the pressure variation along the streamline from point A far in front the sphere $(x_A = -\infty$ and $V_A = V_0)$ to point B on the sphere $(x_B = -a$ and $V_B = 0)$



(a)



(b)



■ **Figure E3.1**

Solution:

Since the flow is steady and inviscid, Eq. 3.4 is valid. In addition, since the streamline is horizontal, $\sin\theta = \sin 0 = 0$ and the equation of motion along the streamline reduces to

$$\frac{\partial p}{\partial s} = -\rho V \frac{\partial V}{\partial s} \qquad (1)$$

with the given velocity variation along the streamline, the acceleration term is $V\frac{\partial V}{\partial s} = V\frac{\partial V}{\partial x} = V_0(1+\frac{a^3}{x^3})(-\frac{3V_0a^3}{x^2})$

$$= -3V_0^2(1+\frac{a^3}{x^3})\frac{a^3}{x^4}$$

where we have replaced s by x since the two coordinates are identical (within an additive constant) along streamline A-B. It follows that $V \partial V / \partial s < 0$ along the streamline. The fluid slows down from V_0 far ahead of the sphere to zero velocity on the “nose” of the sphere $(x = -a)$. Thus according to Eq. 1, to produce the given motion the pressure gradient along the streamline is $\dfrac{\partial p}{\partial x} = \dfrac{3\rho a^3 V_0^2 (1 + a^3 / x^3)}{x^4}$ (2)

This variation is indicated in Fig. E3. 1b. It is seen that the pressure increase in the direction of flow $(\partial p / \partial x > 0)$ from point A to point B. The maximum pressure gradient $(0.610\, \rho \mathrm{V}_0^2 / a)$ occurs just slightly ahead of the sphere $(x = -1.205a)$. It is the pressure gradient that slows the fluid down from $V_A = V_0$ to $V_B = 0$.

The pressure distribution along the streamline can be obtained by integrating Eq. 2 from $p = 0$ (gage) at $x = -\infty$ to pressure p at location x. The result, plotted in Fig. E3.1c, is

$$p = -\rho V_0^2 \left[\left(\frac{a}{x}\right)^3 + \frac{(a/x)^6}{2} \right] \qquad \text{(Ans)}$$

The pressure at B, a stagnation point since $V_B = 0$, is the highest pressure along the streamline $(p_B = \rho V_0^2 / 2)$. As shown in Chapter 9, this excess pressure on the front of the sphere (i.e. $p_B > 0$) contributes to the net drag force on the sphere. Note that the pressure gradient and pressure are directly proportional to the density of the fluid, a representation of the fact that the fluid inertia is proportional to its mass.

Since $\sin\theta = \dfrac{dz}{ds}$, also $V\dfrac{dV}{ds} = \dfrac{1}{2}\dfrac{d(V^2)}{ds}$, and along the streamline, $n = const$, so $dn = 0$.

Thus

$$dp = \frac{\partial p}{\partial s}ds + \frac{\partial p}{\partial n}dn = \frac{dp}{ds}ds$$

Eq. (3.4) $\rightarrow -\gamma\dfrac{dz}{ds} - \dfrac{dp}{ds} = \dfrac{1}{2}\rho\dfrac{d(V^2)}{ds}$

along a streamline (s)

$$dp + \frac{1}{2}\rho d(V^2) + \gamma dz = 0 \qquad (3.5)$$

or $\displaystyle\int\frac{dp}{\rho} + \frac{1}{2}V^2 + gz = const$ (3.6)

Note: if $\rho \neq \text{constant}$, then $\rho = f(p)$ must be known.

The well-known Bernoulli equation:

$$P+\frac{1}{2}\rho V^2+\gamma Z=\text{constant}$$

Note: 4 assumptions

1) $\mu=0$ (inviscid)

2) $\frac{\partial}{\partial t}=0$ (steady)

3) $\rho=C$ (incompressible)

4) along a streamline ($dn=0$)

Example 3.2 Consider the flow of air around a bicyclist moving through still air with velocity V_0 as is shown in Fig. E3.2. Determine the difference in the pressure between points (1) and (2).

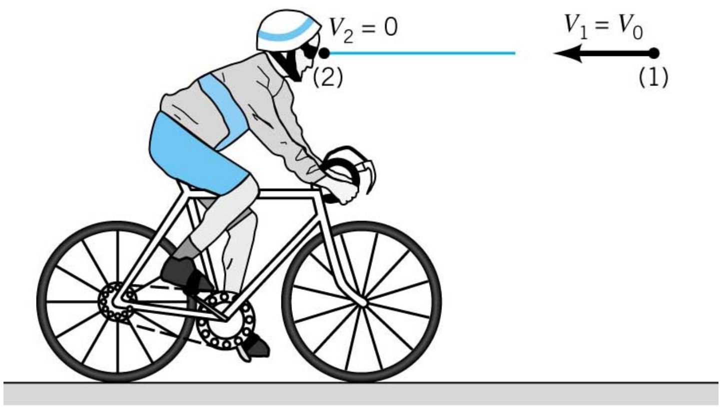


Figure E 3.2

Solution: In a coordinate system fixed to the bike, it appears as through the air is flowing steadily toward the bicycle with speed V_0. If the assumptions of Bernoulli's equation are valid (steady, incompressible, inviscid flow), Eq. 3.7 can be applied as follows along the streamline that passes through (1) and (2)

$$p_1 + \frac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \frac{1}{2}\rho V_2^2 + \gamma z_2$$

We consider (1) to be in the free stream so that $V_1 = V_0$ and (2) to be at the tip of the bicyclist's nose and assume that $z_1 = z_2$ and $V_2 = 0$ (both of which, as is discussed in Section 3.4, are reasonable assumptions). It follows that the pressure at (2) is greater than at (1) by an amount

$$p_2 - p_1 = \frac{1}{2}\rho V_1^2 = \frac{1}{2}\rho V_0^2 \qquad \text{(Ans)}$$

A similar result was obtained in Example 3.1 by integrating pressure gradient, which was known because the velocity distribution along the streamline, $V(s)$, was known. The Bernoulli equation is a general integration of $F = ma$. To determine $p_2 - p_1$, knowledge of the detailed velocity distribution is not needed-only the "boundary conditions" at (1) and (2) are required. Of course, knowledge of the value of V along the streamline is needed to determine the speed V_0. As discussed in Section 3.5, this is the principle upon which many velocity measuring devices are based.

If the bicyclist were accelerating or decelerating, the flow would unsteady (i.e. $V_0 \neq \text{constant}$) and the above analysis would be in correct since Eq. 3.7 is restricted to steady flow.

3.3 $\vec{F} = m\vec{a}$ normal to a streamline

. Example: The devastating low-pressure region at the center of a tornado can be explained by applying Newton's 2nd law across the nearly circular streamlines of the tornado.

. $\delta Fn = \dfrac{\delta m V^2}{R} = \rho\delta \dfrac{V^2}{R} = \delta m|\vec{a}_n|$ (3.8)

. $\delta Wn = -\delta W \cos\theta = -\gamma\delta\forall \cos\theta$ $(\theta = 90°; \delta W_{\text{n}} = 0)$

. $\delta Fpn = (P - \delta Pn)\delta S\delta y - (P + \delta Pn)\delta S\delta y$

$$= -2\delta Pn\delta S\delta y = -\frac{\partial P}{\partial n}\delta S\delta n\delta y$$

$$= -\frac{\partial P}{\partial n}\delta\forall$$

. Thus, $\delta Fn = \delta Wn + \delta Fpn = (-\gamma\cos\theta - \dfrac{\partial P}{\partial n})\delta\forall$ (3.9)

Eq. (3.8) & (3.9) & $\cos\theta = \dfrac{dz}{dn}$

$$\therefore -\gamma\frac{dz}{dn} - \frac{\partial P}{\partial n} = \frac{\rho V^2}{R} \qquad (3.10)$$

. if gravity is neglected or if the floor is horizontal

$$\gamma = 0 \qquad\qquad \frac{dz}{dn} = 0$$

$$\Rightarrow \frac{\partial P}{\partial n} = -\frac{\rho V^2}{R}$$

This pressure difference is needed to be balance the centrifugal acceleration associated with the curved streamlines of the fluid motion.

. Read example 3.3

. Integrate across the streamline, using the factor that

$\partial P / \partial n = dP / dn$ if $S = \text{constant}$

$$\int \frac{dP}{\rho} + \int \frac{V^2}{R} dn + gz = \text{constant} \qquad (3.11)$$

across the streamline

$\vec{V} = \vec{V}(S, n) \quad ; \; R = R(S, n)$

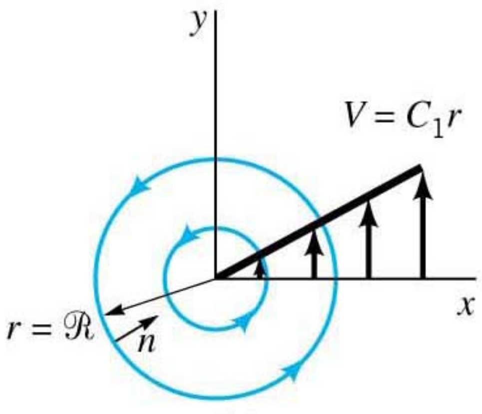
$$\boxed{\begin{array}{l} p + \rho \int \frac{V^2}{R} dn + \gamma z = C \qquad (3.12) \\ \text{if } \; 1)\ \frac{\partial}{\partial t} \quad 2)\ \mu = 0 \quad 3)\ \rho = C \end{array}}$$

Example 3.3 Shown in Figs. E3.3a, b are two fields with circular streamlines. The velocity distributions are

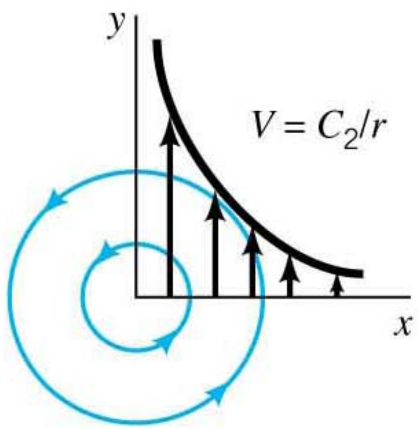
$$V(r) = C_1 r \quad \text{for case (a)} \quad \text{and}$$

$$V(r) = \frac{C_2}{r} \quad \text{for case(b)}$$

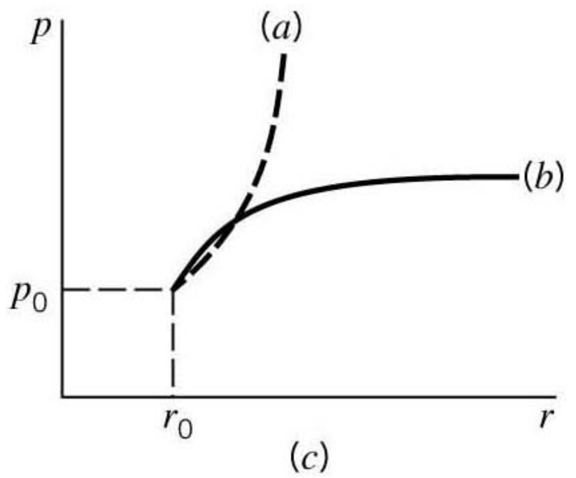
where C_1 and C_2 are constant. Determine the pressure distributions, $p = p(r)$, for each given that $p = p_0$ at $r = r_0$.



(a)



(b)



■ **Figure E 3.3**

Solution: We assume the flow are steady, inviscid, and incompressible with streamline in the horizontal plane ($dz/dn = 0$) Since the streamlines are circles, the coordinate n points in a direction opposite of that of the radial coordinate, $\partial/\partial n = -\partial/\partial r$, and the radius of curvature is given by $R = r$. Hence, Eq. 3.10 becomes

$$\frac{\partial p}{\partial r} = \frac{\rho V^2}{r}$$

For case (a) this gives $\frac{\partial p}{\partial r} = \rho C_1^2 r$

While for case (b) it gives $\frac{\partial p}{\partial r} = \frac{\rho C_2^2}{r^3}$

For either the pressure increases as r increase since $\delta p/\delta r > 0$

Integration of these equations with respect to r, starting with a known pressure $p = p_0$ at $r = r_0$, gives

$$p = \frac{1}{2}\rho C_1^2 (r^2 - r_0^2) + p_0 \qquad \text{(Ans)}$$

for case (a) and

$$p = \frac{1}{2}\rho C_2^2 (\frac{1}{r_0^2} - \frac{1}{r^2}) + p_0 \qquad \text{(Ans)}$$

for case (b). These pressure distributions are sketched in Fig E3.3c. The pressure distributions needed to balance the centrifugal accelerations in cases (a) and (b) are not the same because the velocity distributions are different. In fact for case (a) the pressure increase without bound as $r \to \infty$, while for case (b) the pressure approaches a finite value as $r \to \infty$. The streamline patterns are the same for each case, however. Physically, case (a) represents rigid body rotation (as obtained in a can of water on a turntable after it has been “spun up”) and case(b) represents a free vortex (an approximation to a tornado or the swirl of water in a drain, the “bathtub vortex”).

3.4	**Physical Interpretation**

- Along a streamline: $P + \frac{1}{2}\rho V^2 + \gamma z = C$

  Across the streamline: $P + \rho \int \frac{V^2}{R} dn + \gamma z = c$

  If: steady, inviscid, and incompressible (None is exactly true for real flows)

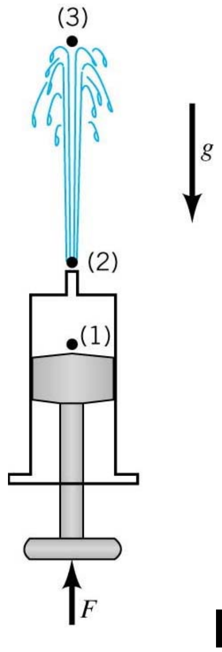
- Physics: Force balance; Bernoulli's Principle

  → The work done on a particle by all forces acting on the particle is equal to the change of the kinetic energy of the particle.

  (Newton's second law; first & second laws of thermodynamics).

Example 3.4

Consider the flow of water from the syringe shown in Fig. 3.4. A force applied to the plunger will produce a pressure greater than atmospheric at point (1) within the syringe. The water flows from the needle, point (2), with relatively high velocity and coasts up to point (3) at the top of its trajectory. Discuss the energy of the fluid at points (1), (2), and (3) by using the Bernoulli equation.



■ Figure E 3.4

Solution: If the assumptions (steady, inviscid, incompressible flow) of the Bernoulli equation are approximately valid, it then follows that the flow can be explained in terms of the partition of the total energy of the water. According to Eq. 3.13 the sum of the three types of energy (kinetic, potential, and pressure) or heads (velocity, elevation, and pressure) must remain constant. The following table indicates the relative magnitude of each of these energies at the three points shown in the figure.

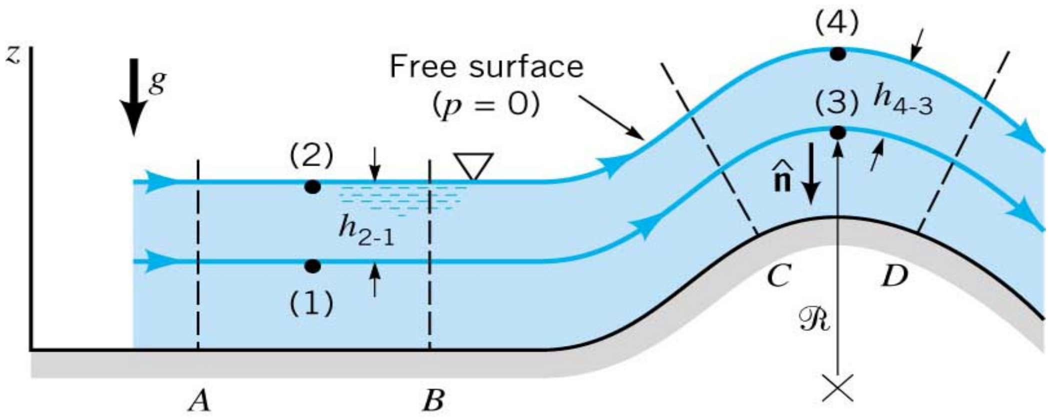
	Energy Type		
	Kinetic	**Potential**	**Pressure**
Point	$\rho V^2/2$	γz	p
1	Small	Zero	Large
2	Large	Small	Zero
3	Zero	Large	Zero

The motion results in (or is due to) a change in the magnitude of each type of energy as the fluid flows from one location to another. An alternate way to consider this flow is as follows. The pressure gradient between (1) and (2) produces an acceleration to eject the water from the needle. Gravity acting on the particle between (2) and (3) produces a deceleration to cause the water to come to a momentary stop at the top of its flight.

If friction (viscous) effects were important, there would be an energy loss between (1) and (3) and for the given p_1 the water would not be able to reach the height indicated in the figure. Such friction may arise in the needle (see chapter 8, pipe flow) or between the water stream and the surrounding air (see chapter

9, external flow).

Example 3.5 Consider the inviscid, incompressible, steady flow shown in Fig. E3.5. From section A to B the streamlines are straight, while from C to D they follow circular paths. Describe the pressure variation between points (1) and (2) and points (3) and (4).



■ **Figure E 3.5**

Solution: With the above assumption and the fact that $R = \infty$ for the portion from *A* to *B*, Eq. 3.14 becomes

$$p + \gamma z = \text{constant}$$

The constant can be determined by evaluating the known variables at the two locations using $P_2 = 0$ (gage), $Z_1 = h_{2\text{-}1}$ to give

$$p_1 = p_2 + \gamma(z_2 - z_1) = p_2 + \gamma h_{2-1} \quad \text{(Ans)}$$

Note that since the radius of curvature of the streamline is infinite, the pressure variation in the vertical direction is the same as if the fluid were stationary. However, if we apply Eq. 3.14 between points (3) band point (4) we obtain (using dn = -dz)

$$p_4 + \rho \int_{Z_3}^{Z_4} \frac{V^2}{R}(-dz) + \gamma z_4 = p_3 + \gamma z_3$$

With p_4=0 and z_4-z_3=h_{4-3} this becomes

$$p_3 = \gamma h_{4-3} - \rho \int_{Z_3}^{Z_4} \frac{V^2}{R} dz \qquad \text{(Ans)}$$

To evaluate the integral we must know the variation of V and R with z. Even without this detailed information we note that the integral has a positive value. Thus, the pressure at (3) is less than the hydrostatic value, γh_{4-3}, by an amount equal to $\rho \int_{Z_3}^{Z_4} \frac{V^2}{R} dz$. This lower pressure, caused by the curved streamline, is necessary to accelerate the fluid around the curved path.

Note that we did not apply the Bernoulli equation (Eq. 3.13) across the streamlines from (1) to (2) or (3) to (4). Rather we used Eq. 3.14. As is discussed in section 3.6 application of the Bernoulli equation across streamlines (rather than along) them may lead to serious errors.

3.5	Static, Stagnation (Total), and Dynamic Pressure

- γz **-** hydrostatic pressure not a real pressure, but possible due to potential energy variations of the fluid as a result of elevation changes.
- ***p*** **-** static pressure
- $\frac{1}{2}\rho V^2$ **-**dynamic pressure
- stagnation point $\rightarrow V = 0$

$$P_2 = P_1 + \frac{1}{2}\rho_1 V_1^2$$

- stagnation pressure $\rightarrow$ static pressure+dynamic pressure (if elevation effect are neglected)

Total pressure

It represents the conversion of all of the kinetic energy into a pressure rise.

$$P_T = \gamma Z + P + \frac{1}{2}\rho V^2$$

Total P Hydrostatic P Static P Dynamic P

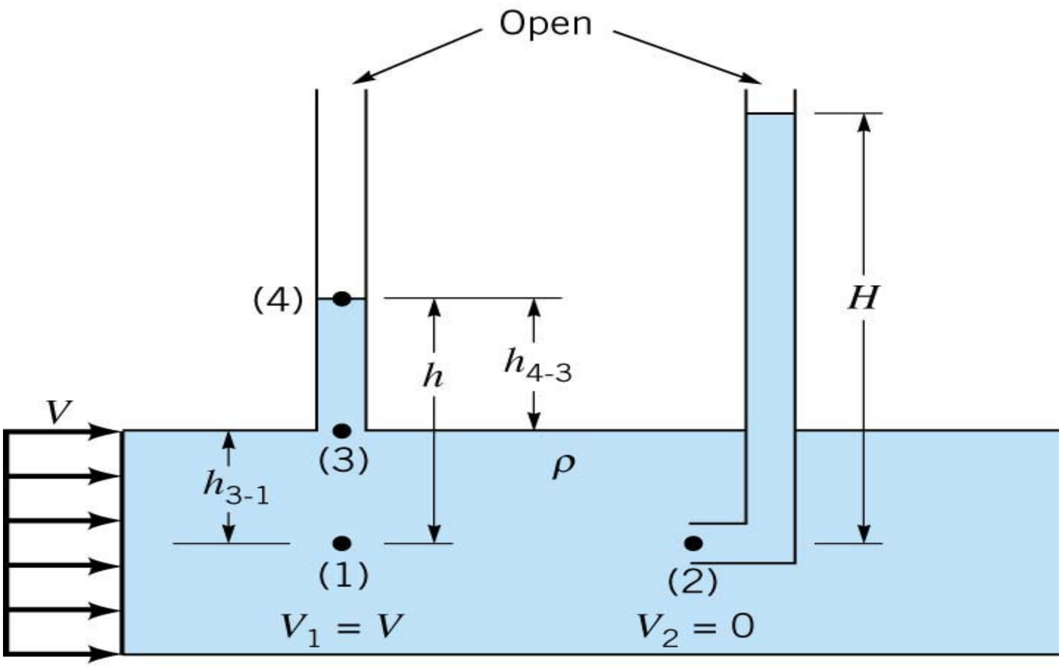
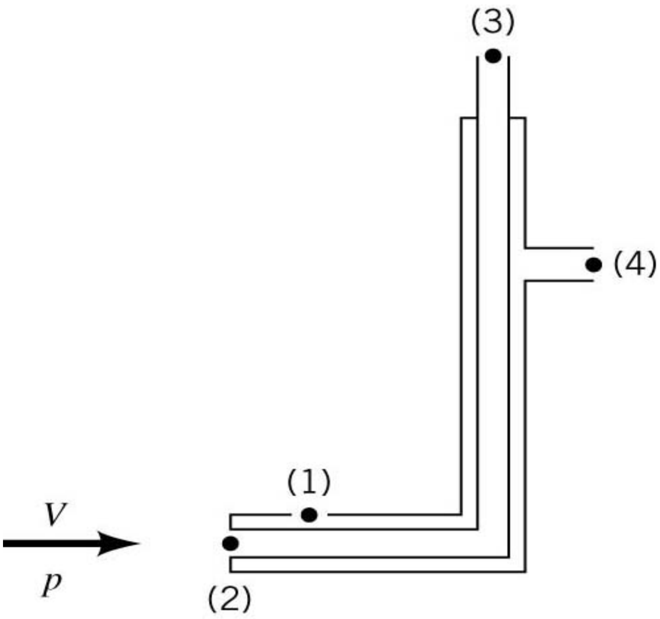
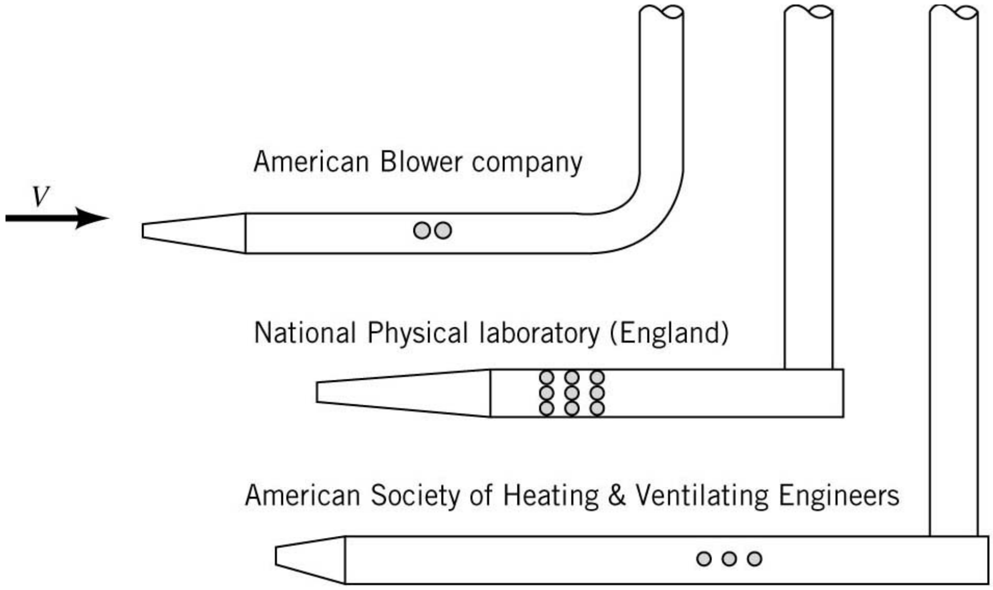


Figure 3.4 Measurement of static and stagnation pressures

• Pitot-static tube: simple, relatively in expensive.



■ **Figure 3.6 The Pitot-static tube**



■ **Figure 3.7 Typical Piotot-static tube designs.**

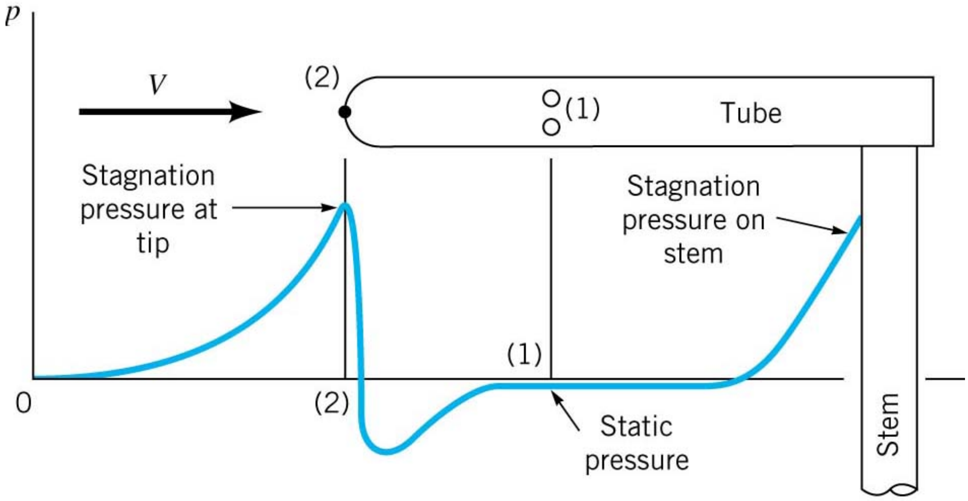
$$p+\frac{1}{2}\rho V^2+\gamma z=p_T=\text{constant} \quad \text{along a streamline}$$

center tube: $p_3=p+\frac{1}{2}\rho V^2$

outer tube: $p_4=p_1=p$

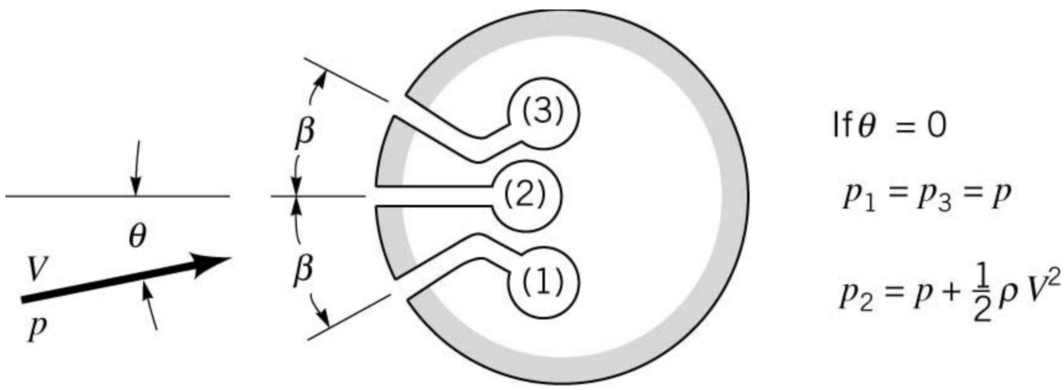
elevation is neglected.

$$p_3-p_4=\frac{1}{2}\rho V^2 \rightarrow \underline{V=\sqrt{2(p_3-p_4)/\rho}}$$



■ **Figure 3.9 Typical pressure distribution along a Pitot-static tube.**

• The cylinder is rotated until the pressures in the two sideholes are equal, thus indicating that the center hole points directly upstream-measure stagnation pressure. Side holes ($\beta = 29.5°$) → measure static pressure.



■ **Figure 3.10 Cross section of a directional-finding Pitot-static tube.**

3.6 Example of Use of the Bernoulli Equation

steady $(\frac{\partial}{\partial t}=0)$, inviscid $(\mu=0)$, incompressible $(\rho=\text{constant})$ and along a streamline $(\frac{\partial}{\partial n}=0)$.

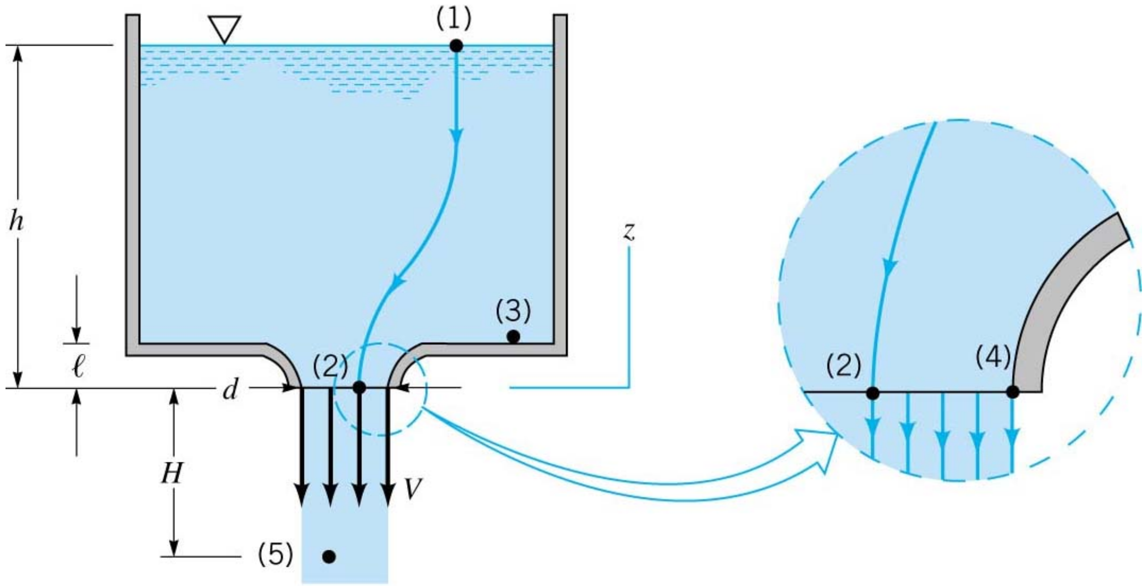
Between two points:

$$\underline{\underline{P_1+\frac{1}{2}\rho V_1^2+\gamma Z_1=P_2+\frac{1}{2}\rho V_2^2+\gamma Z_2}}$$

Unknowns: $P_1, P_2, V_1, V_2, Z_1, Z_2$ (6)

If 5 unknowns are given, then determining the remaining one.

- Free Jets (Vertical flows from a tank)



■ **Figure 3.11 Vertical flow from a tank**

<u>(1) - (2)</u>:

$P_1 = 0$ (gage pressure); $P_2 = 0$ (Free jet)

$Z_1 = h$; $Z_2 = 0$; $V_1 \approx 0$;

$$\Rightarrow \gamma h = \frac{1}{2}\rho V_2^2$$

$$V_2 = \sqrt{\frac{2\gamma h}{\rho}} = \sqrt{2gh}$$

<u>(2) and (4)</u>: atmospheric pressure

<u>(5)</u>: $V_5 = \sqrt{2g(h+H)}$

(3) and (4): $P_3 = \gamma(h-l)$; $P_4 = 0$; $Z_3 = l$; $Z_4 = 0$,

$V_3=0 \longrightarrow P_3 + \gamma l = P_4 + \frac{1}{2}\rho V_4^2 \qquad \because V_4 = \sqrt{2gh}$

- Recall from Physics or dynamics

自由落體在一真空中 $V = \sqrt{2gh}$; the same as the liquid leaving from the nozzle. All potential energy $\rightarrow$ kinetic energy (neglecting the viscous effects).

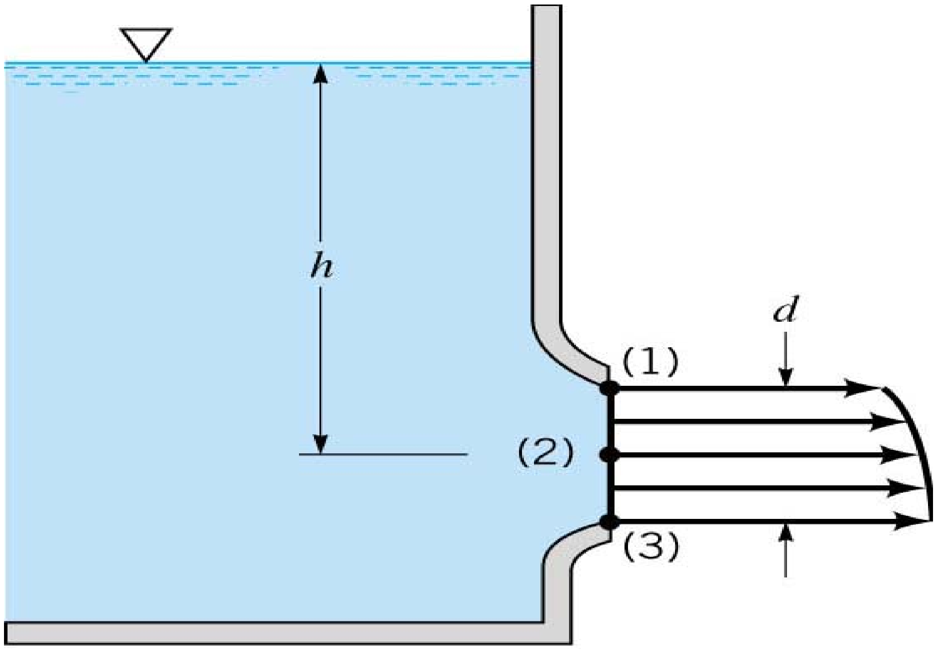


Fig 3.12 Horizontal flow from a tank.

- Vena contracta effect:

Fluid can not turn 90°; Contraction coefficient: $C_c = A_j/A_h$;

$A_j \rightarrow$ cross area of jet fluid column; $A_h \rightarrow$ cross area of the nozzle exit.

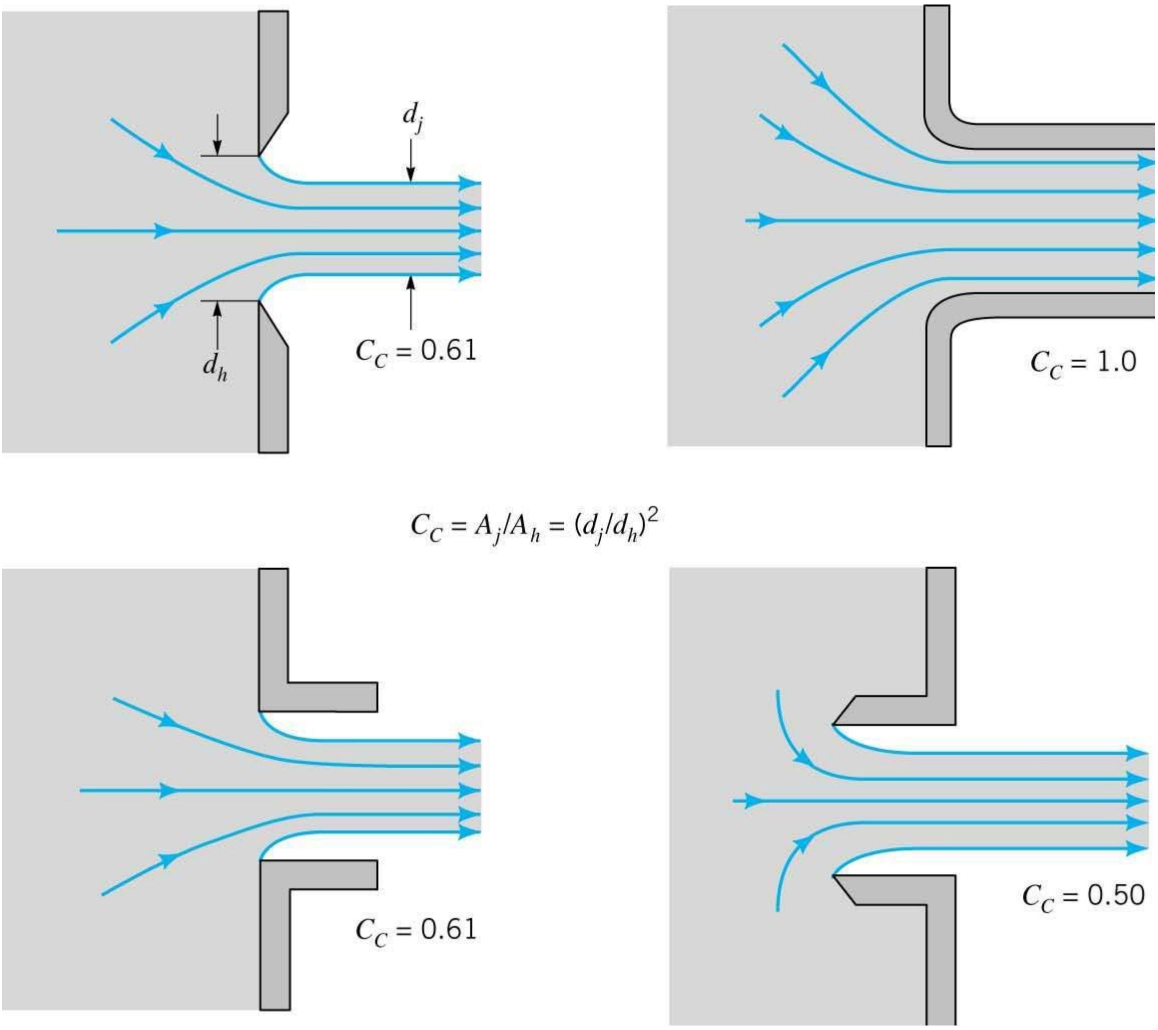
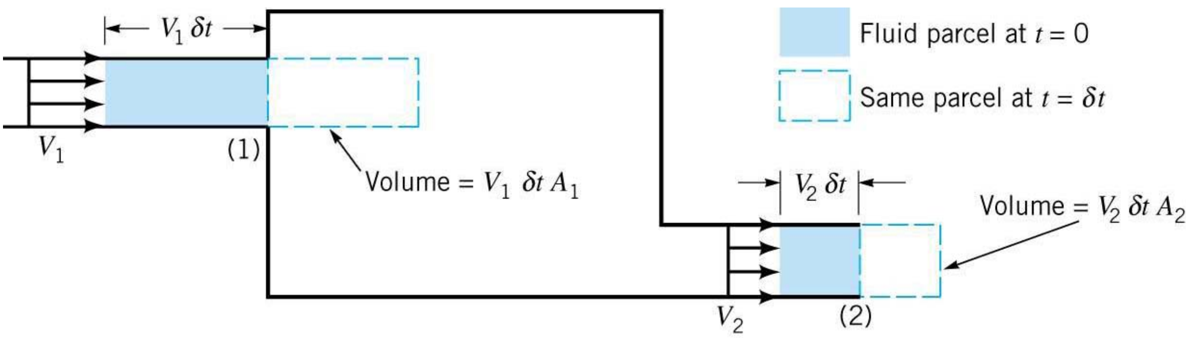


Figure 3.14 Typical flow patterns and contraction coefficients for various round exit configurations.

3.6.2 Confined flows



■ **Figure 3.15 Steady flow into and out of a tank.**

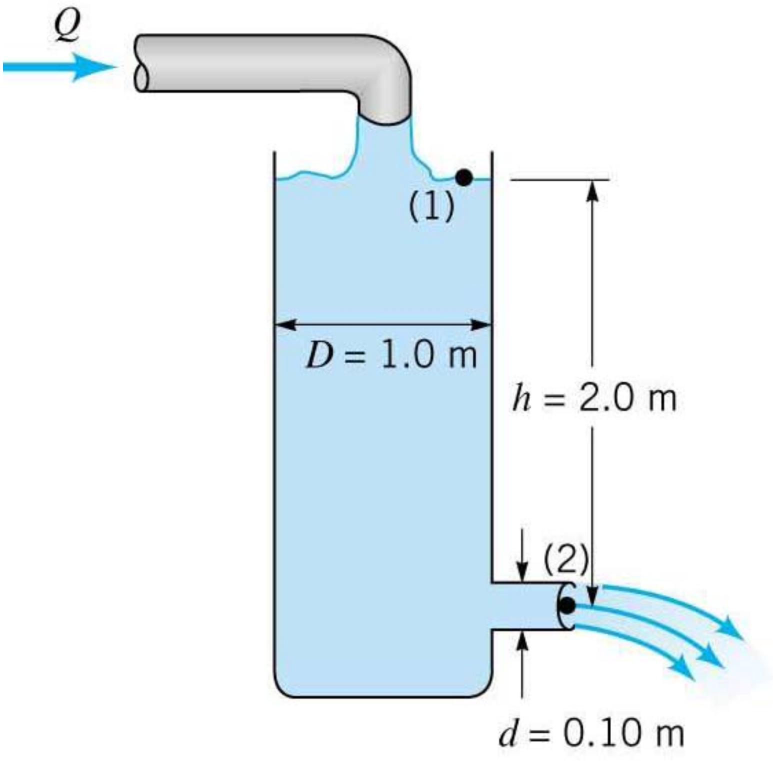
mass flow rate $\dot{m} = \rho \cdot Q$ (kg/s or slugs/s)

where Q: volume flow rate (m^3/s or ft^3/s)

$Q = VA$; $\dot{m} = \rho \cdot Q = \rho VA$

From Fig. 3.15: $\rho_1 A_1 V_1 = \rho_2 A_2 V_2$

Example 3.7 A stream of diameter d = 0.1m flows steadily from a tank of diameter D = 1.0m as shown in Fig E3.7a. Determine the flow rate, Q, needed from the inflow pipe if the water depth remains constant. h = 2.0m.



(a)

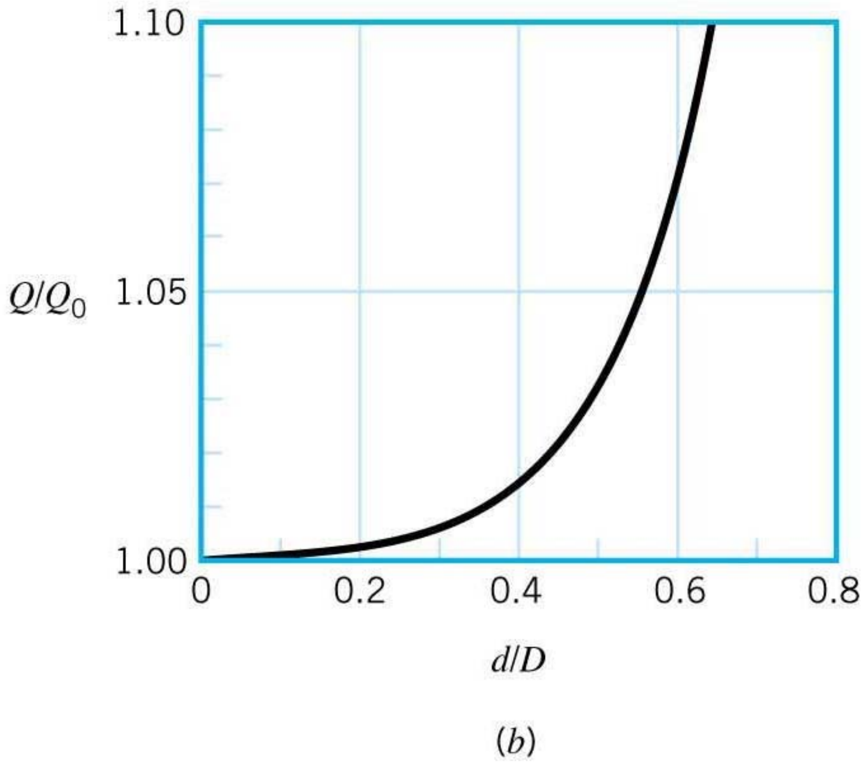


Figure E 3.7

Solution: For steady, inviscid, incompressible the Bernoulli equation applied between points (1) and (2) is

$$p_1 + \frac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \frac{1}{2}\rho V_2^2 + \gamma Z_2 \quad (1)$$

With the assumptions that $p_1 = p_2 = 0$, $z_1 = h$, and $z_2 = 0$, Eq. 1 becomes

$$\frac{1}{2}V_1^2 + gh = \frac{1}{2}V_2^2 \qquad (2)$$

Although the water level remains constant (h = constant), there is an average velocity, V_1, across section (1) because of the flow from the tank. From Eq 3.19 for steady incompressible flow, conservation of mass requires $Q_1 = Q_2 = AV$. Thus, $A_1V_1 = A_2V_2$, or

$$\frac{\pi}{4}D^2V_1 = \frac{\pi}{4}d^2V_2$$

Hence, $V_1 = (\frac{d}{D})^2 V_2$ (3)

Equation 1 and 3 can be combined to give

$$V_2 = \sqrt{\frac{2gh}{1-(d/D)^4}}$$

Thus, with the given data

$$V_2 = \sqrt{\frac{2(9.81\,\mathrm{m/s^2})(2.0\,\mathrm{m})}{1-(0.1\,\mathrm{m}/1\,\mathrm{m})^4}} = 6.26\,\mathrm{m/s}$$

and $Q = A_1V_1 = A_2V_2 = \frac{\pi}{4}(0.1\mathrm{m})^2(6.26\mathrm{m/s}) = 0.0492\mathrm{m^3/s}$ (Ans)

In this example we have not neglected the kinetic energy in the water in the tank ($V_1 \neq 0$). If the tank diameter is large compared to the jet diameter ($D >>> d$), Eq. 3 indicates that $V_1 << V_2$ and the assumption that $V_1 \approx 0$ would be reasonable. The error associated with this assumption can be seen by calculating the ratio of the flowrate assuming $V_1 \neq 0$, denoted Q, to that assuming $V_1 = 0$, denoted Q_0. This ratio, written as

$$\frac{Q}{Q_0}=\frac{V_2}{V_2\big|_{D=\infty}}=\frac{\sqrt{2gh/(1-(d/D)^4)}}{\sqrt{2gh}}=\frac{1}{\sqrt{1-(d/D)^4}}$$

is plotted in Fig. E3.7b. With $0< d/D < 0.4$ it follows that $1< Q/Q_0 \leq 1.01$, and the error in assuming $V_1 = 0$ is less than 1%. Thus, it is often reasonable to assume $V_1 = 0$.

Example 3.9

Water flows through a pipe reducer as is shown in Fig. E3.9. The static pressure at (1) and (2) are measured by the inverted U-tube manometer containing oil of specific gravity. SG. Less than one. Determine the manometer reading, *h*.

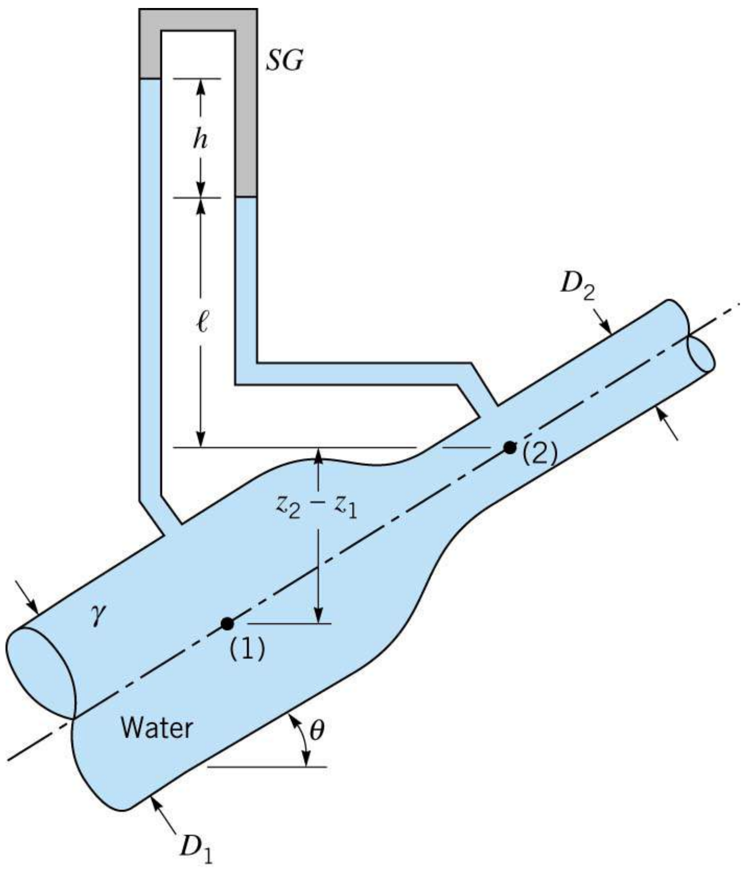


FIGURE E 3.9

Solution: With the assumption of steady, inviscid, incompressible flow, the Bernoulli equation can be written as

$$p_1 + \frac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \frac{1}{2}\rho V_2^2 + \gamma z_2$$

The continuity equation (Eq. 3.19) provides a second relationship between V_1 and $V_{2,}$ if we assume the velocity profiles are uniform as those two locations and the fluid incompressible:

$$Q = A_1 V_1 = A_2 V_2$$

By combining these two equations we obtain

$$p_1 - p_2 = \gamma (z_2 - z_1) + \frac{1}{2}\rho V_2^2 [1 - \left(\frac{A_2}{A_1}\right)^2] \qquad (1)$$

This pressure difference is measured by the manometer and can be determined by using the pressure-depth ideas developed in Chapter 2. Thus.

$$p_1 - \gamma (z_2 - z_1) - \gamma l - \gamma h + SG\gamma h + \gamma l = p_2$$

or $$p_1 - p_2 = \gamma (z_2 - z_1) + (1 - SG)\gamma h \qquad (2)$$

As discussed in Chapter 2, the pressure difference is neither merely γh not $\gamma (h+z_1-z_2)$.

Equations 1 and 2 can be combined to give the desired result as follows $$(1 - SG)\gamma h = \frac{1}{2}\rho V_2^2 [1 - \left(\frac{A_2}{A_1}\right)^2]$$

or $V_2 = Q/A_2$, thus, $h = \left(\dfrac{Q}{A_2}\right)^2 \dfrac{1-\left(\dfrac{A_2}{A_1}\right)^2}{2g(1-SG)}.$ (Ans)

The difference in elevation, z_1-z_2, was not needed because the change in elevation term in the Bernoulli equation exactly cancels the elevation term in the manometer equation. However, the pressure difference, p_1-p_2, depends on the angle θ, because of the elevation, z_1-z_2, in Eq. 1. Thus, for a given flowrate, the pressure difference, p_1-p_2, as measured by a pressure gage would vary with θ, but the manometer reading, h, would be independent of θ.

In general, an increase in velocity is accompanied by a decrease in pressure.

Air (gases): Compressibility (Ch 11)

Liquids: Cavitation (Propeller etc.)

3.6.3 Flow Rate Measurement

- Ideal flow meter — neglecting viscous and compressible effects. (loss some accuracy)

- Orifice meter; Nozzle meter: $V \uparrow$, $P \downarrow$; Venturi meter

$$Q = A_2 \sqrt{\frac{2(P_2 - P_1)}{\rho\left[1-\left(\frac{A_2}{A_1}\right)^2\right]}} \tag{3.20}$$

- Sluice gate: $Q = z_2 b \sqrt{\frac{2g(z_2 - z_1)}{1-\left(\frac{z_2}{z_1}\right)^2}}$ (3.21)

- Sharp crested weir: $Q = C_1 H b\sqrt{2gH} = C_1 b\sqrt{2g}H^{\frac{3}{2}}$

3.7 The Energy Line and the Hydraulic Grade Line

- Bernoulli equation is an energy equation with four assumptions: inviscid, incompressible, steady flow, and along a streamline from pts (1) - (2), respectively, and the sum of partition of energy remains constant from pts (1)-(2).

(Head) $\underline{H = \frac{p}{\gamma} + \frac{V^2}{2g} + z}$ = constant along a streamline.

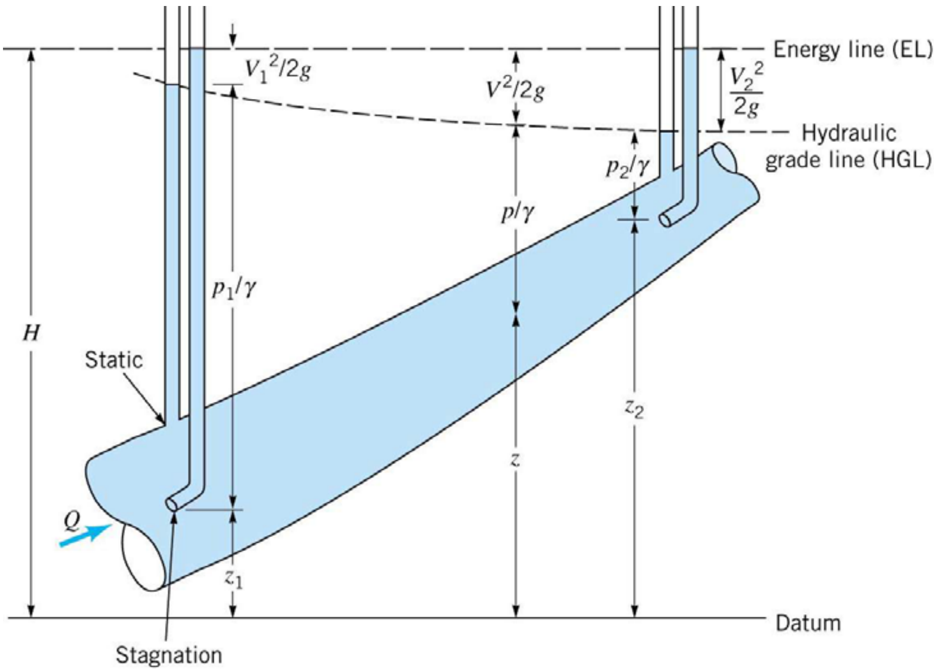


FIGURE 3.21 Representation of the energy line and the hydraulic grade line

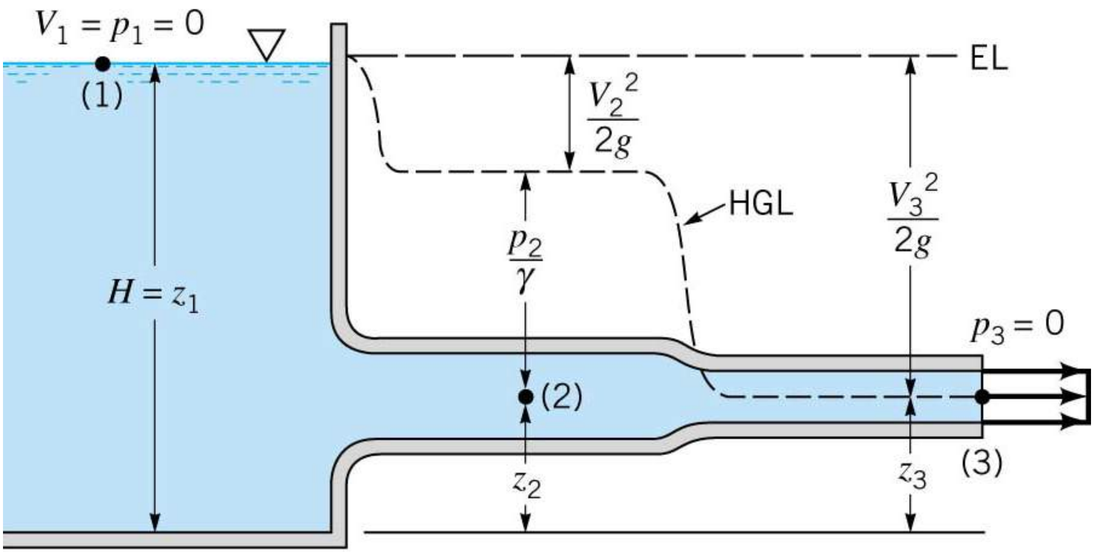


FIGURE 3.22 The energy line and hydraulic grade line for flow from a tank.

3.8 Restrictions on the use of Bernoulli Equation

3.8.1 Compressibility Effect

Gases: $\int \frac{dp}{\rho}$ not a constant

A special case: for compressible flow

Given - steady, inviscid, isothermal ($T = C$ along the streamline)

Sol: $p = \rho RT \rightarrow \rho = \frac{p}{RT}$

$$\int \frac{dp}{\rho} = RT \int \frac{dp}{p} = RT\, ln\, p + c$$

$$\therefore \frac{V_1^2}{2g} + z_1 = \frac{V_2^2}{2g} + z_2 + \frac{RT}{g} ln\left(\frac{p_2}{p_1}\right)$$

or $\frac{V_1^2}{2g} + z_1 + \frac{RT}{g} ln\left(\frac{p_1}{p_2}\right) = \frac{V_2^2}{2g} + z_2$

Compare to the incompressible Bernoulli Equation.

$$\because if\ \frac{p_1}{p_2} = 1 + \frac{p_1 - p_2}{p_2} = 1 + \varepsilon\ ,\ \varepsilon = \frac{p_1 - p_2}{p_2} << 1$$

then $\ln\left(\frac{p_1}{p_2}\right) = \ln(1+\varepsilon) \approx \varepsilon = \frac{p_1 - p_2}{p_2} = \frac{p_1}{p_2} - 1$

$$\frac{V_1^2}{2g} + z_1 + \frac{RT}{g}\left(\frac{p_1}{p_2} - 1\right) = \frac{V_2^2}{2g} + z_2$$

$$\therefore \frac{V_1^2}{2g} + z_1 + \frac{p_1}{\gamma} = \frac{V_2^2}{2g} + z_2 + \frac{p_2}{\gamma}$$

3.8.2 Unsteady Effect

$V = V(s)$ steady; $\because V = V(s, t)$ unsteady.

$$a_s = \frac{\partial V}{\partial t} + V\frac{\partial V}{\partial s}$$ (local + convective accelerations)

- Repeating the steps leading to the Bernoulli Equation.

$$\rho \frac{\partial \mathrm{V}}{\partial \mathrm{t}} ds + dp + \frac{1}{2}\rho\, d\left(V^2\right) + \gamma\, dz = 0$$

along a streamline

- If incompressible, inviscid,

$$p_1 + \frac{1}{2}\rho_1 V_1^2 + \gamma z_1 = \rho \int_{S_1}^{S_2} \frac{\partial v}{\partial t} ds + p_2 + \frac{1}{2}\rho V_2^2 + \gamma z_2$$

Use velocity potential to simplify the problem (Ch 6).

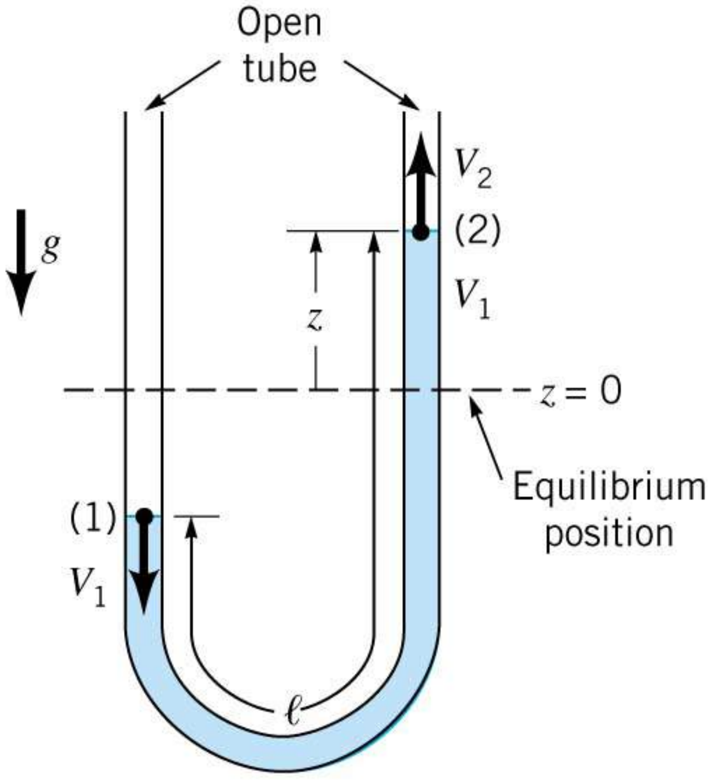


FIGURE 3.25 Oscillation of a liquid column in a U-tube

3.8.3 Rotational Effect

In general, the Bernoulli constant varies from streamline to streamline. However, under certain restrictions, this constant may be the same throughout the entire flow field, as ex.3.19 illustrates this effect.

3.8.4 Other restrictions

$\mu = 0$ (inviscid); Bernoulli Equation: A first integral of Newton's 2^{nd} laws along a streamline.