421-316 Engineering Hydraulics and Hydrology
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Open channel hydraulics

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Abstract

This course of 15 lectures provides an introduction to open channel hydraulics, the generic name for the study of flows in rivers, canals, and sewers, where the distinguishing characteristic is that the surface is unconfined. This means that the location of the surface is also part of the problem, and allows for the existence of waves – generally making things more interesting!

At the conclusion of this subject students will understand the nature of flows and waves in open channels and be capable of solving a wide range of commonly encountered problems.

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Useful references

The following table shows some of the many references available, which the lecturer may refer to in these notes, or which students might find useful for further reading. For most books in the list, The University of Melbourne Engineering Library Reference Numbers are given.

Reference	**Comments**
Bos, M. G. (1978), *Discharge Measurement Structures*, second edn, International Institute for Land Reclamation and Improvement, Wageningen.	Good encyclopaedic treatment of structures
Bos, M. G., Replogle, J. A. & Clemmens, A. J. (1984), *Flow Measuring Flumes for Open Channel Systems*, Wiley.	Good encyclopaedic treatment of structures
Chanson, H. (1999), *The Hydraulics of Open Channel Flow*, Arnold, London.	Good technical book, moderate level, also sediment aspects
Chaudhry, M. H. (1993), *Open-channel flow*, Prentice-Hall.	Good technical book
Chow, V. T. (1959), *Open-channel Hydraulics*, McGraw-Hill, New York.	Classic, now dated, not so readable
Dooge, J. C. I. (1992) , The Manning formula in context, *in* B. C. Yen, ed., *Channel Flow Resistance: Centennial of Manning's Formula*, Water Resources Publications, Littleton, Colorado, pp. 136–185.	Interesting history of Manning's law
Fenton, J. D. & Keller, R. J. (2001), The calculation of streamflow from measurements of stage, Technical Report 01/6, Co-operative Research Centre for Catchment Hydrology, Monash University.	Two level treatment - practical aspects plus high level review of theory
Francis, J. & Minton, P. (1984), *Civil Engineering Hydraulics*, fifth edn, Arnold, London.	Good elementary introduction
French, R. H. (1985), *Open-Channel Hydraulics*, McGraw-Hill, New York.	Wide general treatment
Henderson, F. M. (1966), *Open Channel Flow*, Macmillan, New York.	Classic, high level, readable
Hicks, D. M. & Mason, P. D. (1991) , *Roughness Characteristics of New Zealand Rivers*, DSIR Marine and Freshwater, Wellington.	Interesting presentation of Manning's n for different streams
Jain, S. C. (2001), *Open-Channel Flow*, Wiley.	High level, but terse and readable
Montes, S. (1998), *Hydraulics of Open Channel Flow*, ASCE, New York.	Encyclopaedic
Novak, P., Moffat, A. I. B., Nalluri, C. & Narayanan, R. (2001), *Hydraulic Structures*, third edn, Spon, London.	Standard readable presentation of structures
Townson, J. M. (1991), *Free-surface Hydraulics*, Unwin Hyman, London.	Simple, readable, mathematical

1. Introduction

The flow of water with an unconfined free surface at atmospheric pressure presents some of the most common problems of fluid mechanics to civil and environmental engineers. Rivers, canals, drainage canals, floods, and sewers provide a number of important applications which have led to the theories and methods of open channel hydraulics. The main distinguishing characteristic of such studies is that the location of the surface is also part of the problem. This allows the existence of waves, both stationary and travelling. In most cases, where the waterway is much longer than it is wide or deep, it is possible to treat the problem as an essentially one-dimensional one, and a number of simple and powerful methods have been developed.

In this course we attempt a slightly more general view than is customary, where we allow for real fluid effects as much as possible by allowing for the variation of velocity over the waterway cross section. We recognise that we can treat this approximately, but it remains an often-unknown aspect of each problem. This reminds us that we are obtaining approximate solutions to approximate problems, but it does allow some simplifications to be made.

The basic approximation in open channel hydraulics, which is usually a very good one, is that variation along the channel is gradual. One of the most important consequences of this is that the pressure in the water is given by the hydrostatic approximation, that it is proportional to the depth of water above.

In Australia there is a slightly non-standard nomenclature which is often used, namely to use the word ”channel” for a canal, which is a waterway which is usually constructed, and with a uniform section. We will use the more international English convention, that such a waterway is called a canal, and we will use the words ”waterway”, ”stream”, or ”channel” as generic terms which can describe any type of irregular river or regular canal or sewer with a free surface.

1.1 Types of channel flow to be studied

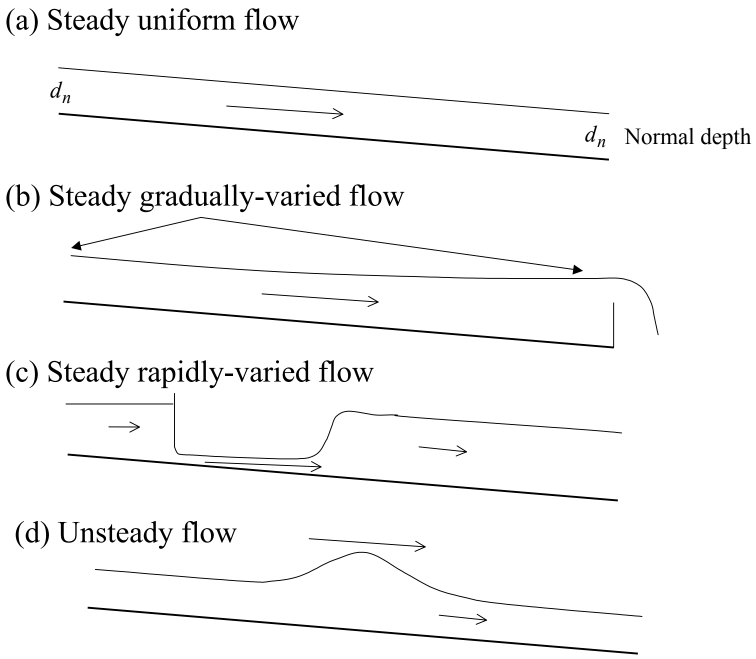

Figure 1-1. Different types of flow in an open channel

Case (a) – Steady uniform flow: Steady flow is where there is no change with time, $\partial/\partial t \equiv 0$. Distant from control structures, gravity and friction are in balance, and if the cross-section is constant, the flow is uniform, $\partial/\partial x \equiv 0$. We will examine empirical laws which predict flow for given bed slope and roughness and channel geometry.

Case (b) – Steady gradually-varied flow: Gravity and friction are in balance here too, but when a control is introduced which imposes a water level at a certain point, the height of the surface varies along the channel for some distance. For this case we will develop the differential equation which describes how conditions vary along the waterway.

Case (c) – Steady rapidly-varied flow: Figure 1-1(c) shows three separate gradually-varied flow states separated by two rapidly-varied regions: (1) flow under a sluice gate and (2) a hydraulic jump. The complete problem as presented in the figure is too difficult for us to study, as the basic hydraulic approximation that variation is gradual and that the pressure distribution is hydrostatic breaks down in the rapid transitions between the different gradually-varied states. We can, however, analyse such problems by considering each of the almost-uniform flow states and consider energy or momentum conservation between them as appropriate. In these sorts of problems we will assume that the slope of the stream balances the friction losses and we treat such problems as frictionless flow over a generally-horizontal bed, so that for the individual states between rapidly-varied regions we usually consider the flow to be uniform and frictionless, so that the whole problem is modelled as a sequence of quasi-uniform flow states.

Case (d) – Unsteady flow: Here conditions vary with time and position as a wave traverses the waterway. We will obtain some results for this problem too.

1.2 Properties of channel flow

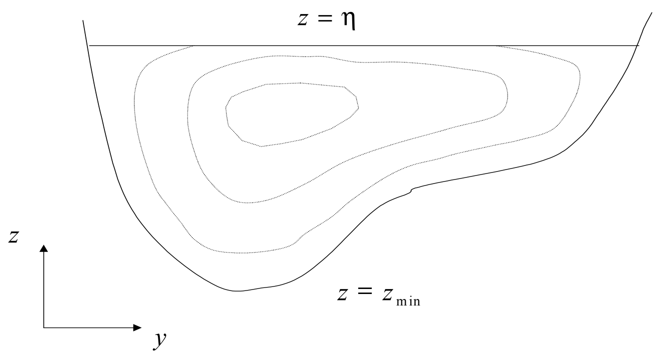

Figure 1-2. Cross-section of flow, showing *isovels*, contours on which velocity normal to the section is constant.

Consider a section of a waterway of arbitrary section, as shown in Figure 1-2. The x co-ordinate is horizontal along the direction of the waterway (normal to the page), y is transverse, and z is vertical. At the section shown the free surface is $z = \eta$, which we have shown to be horizontal across the section, which is a good approximation in many flows.

1.2.1 Discharge across a cross-section

The volume flux or discharge Q at any point is

$$Q = \int_A u \, dA = UA$$

where u is the velocity component in the x or downstream direction, and A is the cross-sectional area. This equation defines the mean horizontal velocity over the section U. In most hydraulic applications the discharge is a more important quantity than the velocity, as it is the volume of water and its rate of propagation, the discharge, which are important.

1.2.2 A generalisation – net discharge across a control surface

Having obtained the expression for volume flux across a plane surface where the velocity vector is normal to the surface, we introduce a generalisation to a control volume of arbitrary shape bounded by a control surface CS. If $\mathbf{u}$ is the velocity vector at any point throughout the control volume and $\hat{\mathbf{n}}$ is a unit vector with direction normal to and directed outwards from a point on the control surface, then $\mathbf{u} \cdot \hat{\mathbf{n}}$ on the control surface is the component of velocity normal to the control surface. If dS is an elemental area of the control surface, then the rate at which fluid volume is leaving across the control surface over that elemental area is $\mathbf{u}\cdot\hat{\mathbf{n}}\,dS$, and integrating gives

$$\text{Total rate at which fluid volume is leaving across the control surface} = \int_{\mathrm{CS}} \mathbf{u}\cdot\hat{\mathbf{n}}\,dS. \qquad (1.1)$$

If we consider a finite length of channel as shown in Figure 1-3, with the control surface made up of

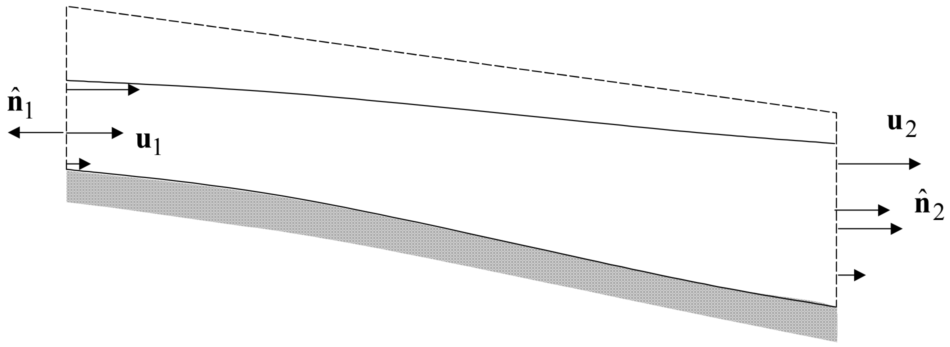

Figure 1-3. Section of waterway and control surface with vertical ends

the bed of the channel, two vertical planes across the channel at stations 1 and 2, and an imaginary enclosing surface somewhere above the water level, then if the channel bed is impermeable, $\mathbf{u}\cdot\hat{\mathbf{n}} = 0$ there; $\mathbf{u} = \mathbf{0}$ on the upper surface; on the left (upstream) vertical plane $\mathbf{u}\cdot\hat{\mathbf{n}} = -u_1$, where u_1 is the horizontal component of velocity (which varies across the section); and on the right (downstream) vertical plane $\mathbf{u}\cdot\hat{\mathbf{n}} = +u_2$. Substituting into equation (1.1) we have

$$\begin{aligned}\text{Total rate at which fluid volume is leaving across the control surface} &= -\int_{A_1} u_1\,dA + \int_{A_2} u_2\,dA \\ &= -Q_1 + Q_2.\end{aligned}$$

If the flow is steady and there is no increase of volume inside the control surface, then the total rate of volume leaving is zero and we have $Q_1 = Q_2$.

While that result is obvious, the results for more general situations are not so obvious, and we will generalise this approach to rather more complicated situations – notably where the water surface in the Control Surface *is* changing.

1.2.3 A further generalisation – transport of other quantities across the control surface

We saw that $\mathbf{u}\cdot\hat{\mathbf{n}}\,dS$ is the *volume* flux through an elemental area – if we multiply by fluid density ρ then $\rho\,\mathbf{u}\cdot\hat{\mathbf{n}}\,dS$ is the rate at which fluid *mass* is leaving across an elemental area of the control surface, with a corresponding integral over the whole surface. Mass flux is actually more fundamental than volume flux, for volume is not necessarily conserved in situations such as compressible flow where the density varies. However in most hydraulic engineering applications we can consider volume to be conserved.

Similarly we can compute the rate at which almost any physical quantity, vector or scalar, is being transported across the control surface. For example, multiplying the mass rate of transfer by the fluid velocity $\mathbf{u}$ gives the rate at which fluid *momentum* is leaving across the control surface, $\rho\mathbf{u}\,\mathbf{u}\cdot\hat{\mathbf{n}}\,dS$.

1.2.4 The energy equation in integral form for steady flow

Bernoulli's theorem states that:

In steady, frictionless, incompressible flow, the energy per unit mass $p/\rho + gz + V^2/2$ is constant

along a streamline,

where V is the fluid speed, $V^2 = u^2+v^2+w^2$, in which (u,v,w) are velocity components in a cartesian co-ordinate system (x,y,z) with z vertically upwards, g is gravitational acceleration, p is pressure and ρ is fluid density. In hydraulic engineering it is usually more convenient to divide by g such that we say that the head $p/\rho g + z + V^2/2g$ is constant along a streamline.

In open channel flows (and pipes too, actually, but this seems never to be done) we have to consider the situation where the energy per unit mass varies across the section (the velocity near pipe walls and channel boundaries is smaller than in the middle while pressures and elevations are the same). In this case we cannot apply Bernoulli's theorem across streamlines. Instead, we use an integral form of the energy equation, although almost universally textbooks then neglect variation across the flow and refer to the governing theorem as "Bernoulli". Here we try not to do that.

The energy equation in integral form can be written for a control volume CV bounded by a control surface CS, where there is no heat added or work done on the fluid in the control volume:

$$\underbrace{\frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho\, e\, dV}_{\text{Rate at which energy is increasing inside the CV}} + \underbrace{\int_{\mathrm{CS}} (p+\rho e)\ \mathbf{u}.\hat{\mathbf{n}}\, dS}_{\text{Rate at which energy is leaving the CS}} = 0, \tag{1.2}$$

where t is time, ρ is density, dV is an element of volume, e is the internal energy per unit mass of fluid, which in hydraulics is the sum of potential and kinetic energies

$$e = gz + \frac{1}{2}\left(u^2+v^2+w^2\right),$$

where the velocity vector $\mathbf{u} = (u,v,w)$ in a cartesian coordinate system (x,y,z) with x horizontally along the channel and z upwards, $\hat{\mathbf{n}}$ is a unit vector as above, p is pressure, and dS is an elemental area of the control surface.

Here we consider steady flow so that the first term in equation (1.2) is zero. The equation becomes:

$$\int_{\mathrm{CS}} \left(p + \rho g z + \frac{\rho}{2}\left(u^2+v^2+w^2\right)\right)\ \mathbf{u}.\hat{\mathbf{n}}\, dS = 0.$$

We intend to consider problems such as flows in open channels where there is usually no important contribution from lateral flows so that we only need to consider flow entering across one transverse face of the control surface across a pipe or channel and leaving by another. To do this we have the problem of integrating the contribution over a cross-section denoted by A which we also use as the symbol for the cross-sectional area. When we evaluate the integral over such a section we will take u to be the velocity along the channel, perpendicular to the section, and v and w to be perpendicular to that. The contribution over a section of area A is then $\pm E$, where E is the integral over the cross-section:

$$E = \int_A \left(p + \rho g z + \frac{\rho}{2}\left(u^2+v^2+w^2\right)\right)\ u\, dA, \tag{1.3}$$

and we take the $\pm$ depending on whether the flow is leaving/entering the control surface, because $\mathbf{u}.\hat{\mathbf{n}} = \pm u$. In the case of no losses, E is constant along the channel. The quantity $\rho Q E$ is the total rate of energy transmission across the section.

Now we consider the individual contributions:

(a) Velocity head term $\frac{\rho}{2}\int_A \left(u^2+v^2+w^2\right)\ u\, dA$

If the flow is swirling, then the v and w components will contribute, and if the flow is turbulent there will be extra contributions as well. It seems that the sensible thing to do is to recognise that all velocity components and velocity fluctuations will be of a scale given by the mean flow velocity in the stream at that point,and so we simply write, for the moment ignoring the coefficient $\rho/2$:

$$\int_A \left(u^2+v^2+w^2\right)\, u\, dA = \alpha U^3 A = \alpha \frac{Q^3}{A^2}, \tag{1.4}$$

which defines α as a coefficient which will be somewhat greater than unity, given by

$$\alpha = \frac{\int_A \left(u^2+v^2+w^2\right)\, u\; dA}{U^3 A}. \tag{1.5}$$

Conventional presentations define it as being merely due to the non-uniformity of velocity distribution across the channel:

$$\alpha = \frac{\int_A u^3\, dA}{U^3 A},$$

however we suggest that is more properly written containing the other velocity components (and turbulent contributions as well, ideally). This coefficient is known as a *Coriolis* coefficient, in honour of the French engineer who introduced it.

Most presentations of open channel theory adopt the approximation that there is no variation of velocity over the section, such that it is assumed that $\alpha = 1$, however that is not accurate. Montes (1998, p27) quotes laboratory measurements over a smooth concrete bed giving values of α of 1.035-1.064, while for rougher boundaries such as earth channels larger values are found, such as 1.25 for irrigation canals in southern Chile and 1.35 in the Rhine River. For compound channels very much larger values may be encountered. It would seem desirable to include this parameter in our work, which we will do.

(b) Pressure and potential head terms

These are combined as

$$\int_A (p+\rho g z)\; u\, dA. \tag{1.6}$$

The approximation we now make, common throughout almost all open-channel hydraulics, is the ”hydrostatic approximation”, that pressure at a point of elevation z is given by

$$p \approx \rho g \times \text{ height of water above} = \rho g\,(\eta - z)\,, \tag{1.7}$$

where the free surface directly above has elevation η. This is the expression obtained in hydrostatics for a fluid which is not moving. It is an excellent approximation in open channel hydraulics except where the flow is strongly curved, such as where there are short waves on the flow, or near a structure which disturbs the flow. Substituting equation (1.7) into equation (1.6) gives

$$\rho g \int_A \eta\, u\, dA,$$

for the combination of the pressure and potential head terms. If we make the reasonable assumption that η is constant across the channel the contribution becomes

$$\rho g \eta \int_A u\, dA = \rho g \eta Q,$$

from the definition of discharge Q.

(c) Combined terms

Substituting both that expression and equation (1.4) into (1.3) we obtain

$$E = \rho g Q \left(\eta + \frac{\alpha}{2g}\frac{Q^2}{A^2}\right), \tag{1.8}$$

which, in the absence of losses, would be constant along a channel. This energy flux across entry and exit faces is that which should be calculated, such that it is weighted with respect to the mass flow rate. Most presentations pretend that one can just apply Bernoulli's theorem, which is really only valid along a streamline. However our results in the end are not much different. We can introduce the concept of the *Mean Total Head* H such that

$$H = \frac{\text{Energy flux}}{g \times \text{Mass flux}} = \frac{E}{g \times \rho Q} = \eta + \frac{\alpha}{2g}\frac{Q^2}{A^2}, \tag{1.9}$$

which has units of length and is easily related to elevation in many hydraulic engineering applications, relative to an arbitrary datum. The integral version, equation (1.8), is more fundamental, although in common applications it is simpler to use the mean total head H, which will simply be referred to as the *head* of the flow. Although almost all presentations of open channel hydraulics assume $\alpha = 1$, we will retain the general value, as a better model of the fundamentals of the problem, which is more accurate, but also is a reminder that although we are trying to model reality better, its value is uncertain to a degree, and so are any results we obtain. In this way, it is hoped, we will maintain a sceptical attitude to the application of theory and ensuing results.

(d) Application to a single length of channel – including energy losses

We will represent energy losses by ΔE. For a length of channel where there are no other entry or exit points for fluid, we have

$$E_{\text{out}} = E_{\text{in}} - \Delta E,$$

giving, from equation (1.8):

$$\rho Q_{\text{out}} \left(g\eta + \frac{\alpha}{2}\frac{Q^2}{A^2} \right)_{\text{out}} = \rho Q_{\text{in}} \left(g\eta + \frac{\alpha}{2}\frac{Q^2}{A^2} \right)_{\text{in}} - \Delta E,$$

and as there is no mass entering or leaving, $Q_{\text{out}} = Q_{\text{in}} = Q$, we can divide through by ρQ and by g, as is common in hydraulics:

$$\left(\eta + \frac{\alpha}{2g}\frac{Q^2}{A^2} \right)_{\text{out}} = \left(\eta + \frac{\alpha}{2g}\frac{Q^2}{A^2} \right)_{\text{in}} - \Delta H,$$

where we have written $\Delta E = \rho g Q \times \Delta H$, where ΔH is the head loss. In spite of our attempts to use energy flux, as Q is constant and could be eliminated, in this head form the terms appear as they are used in conventional applications appealing to Bernoulli's theorem, but with the addition of the α coefficients.

2. Conservation of energy in open channel flow

In this section and the following one we examine the state of flow in a channel section by calculating the energy and momentum flux at that section, while ignoring the fact that the flow at that section might be slowly changing. We are essentially assuming that the flow is locally uniform – *i.e.* it is constant along the channel, $\partial/\partial x \equiv 0$. This enables us to solve some problems, at least to a first, approximate, order. We can make useful deductions about the behaviour of flows in different sections, and the effects of gates, hydraulic jumps, *etc.*. Often this sort of analysis is applied to parts of a rather more complicated flow, such as that shown in Figure 1-1(c) above, where a gate converts a deep slow flow to a faster shallow flow but with the same energy flux, and then *via* an hydraulic jump the flow can increase dramatically in depth, losing energy through turbulence but with the same momentum flux.

2.1 The head/elevation diagram and alternative depths of flow

Consider a steady ($\partial/\partial t \equiv 0$) flow where any disturbances are long, such that the pressure is hydrostatic. We make a departure from other presentations. Conventionally (beginning with Bakhmeteff in 1912) they introduce a co-ordinate origin at the bed of the stream and introduce the concept of "specific energy", which is actually the head relative to that special co-ordinate origin. We believe that the use of that datum somehow suggests that the treatment and the results obtained are special in some way. Also, for irregular cross-sections such as in rivers, the ”bed” or lowest point of the section is poorly defined, and we want to minimise our reliance on such a point. Instead, we will use an arbitrary datum for the head, as it is in keeping with other areas of hydraulics and open channel theory.

Over an arbitrary section such as in Figure 1-2, from equation (1.9), the head relative to the datum can be written

$$H = \eta + \frac{\alpha Q^2}{2g} \frac{1}{A^2(\eta)}, \tag{2.1}$$

where we have emphasised that the cross-sectional area for a given section is a known function of surface elevation, such that we write $A(\eta)$. A typical graph showing the dependence of H upon η is shown in Figure 2-1, which has been drawn for a particular cross-section and a constant value of discharge Q, such that the coefficient $\alpha Q^2/2g$ in equation (2.1) is constant.

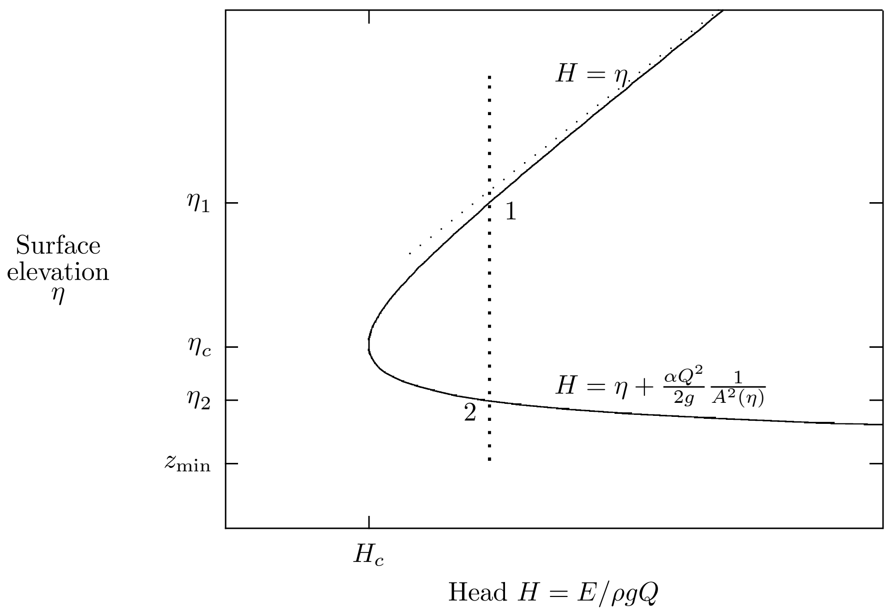

Figure 2-1. Variation of head with surface elevation for a particular cross-section and discharge

The figure has a number of important features, due to the combination of the linear increasing function η and the function $1/A^2(\eta)$ which decreases with η.

- In the shallow flow limit as $\eta \to z_{\min}$ (*i.e.* the depth of flow, and hence the cross-sectional area $A(\eta)$, both go to zero while holding discharge constant) the value of $H \sim \alpha Q^2/2gA^2(\eta)$ becomes very large, and goes to ∞ in the limit.
- In the other limit of deep water, as η becomes large, $H \sim \eta$, as the velocity contribution becomes negligible.
- In between these two limits there is a minimum value of head, at which the flow is called *critical flow*, where the surface elevation is η_c and the head H_c.
- For all other H greater than H_c there are *two* values of depth possible, *i.e.* there are two different flow states possible for the same head.
- The state with the larger depth is called *tranquil*, *slow*, or *sub-critical* flow, where the potential to make waves is relatively small.
- The other state, with smaller depth, of course has faster flow velocity, and is called *shooting*, *fast*, or *super-critical* flow. There is more wave-making potential here, but it is still theoretically possible for the flow to be uniform.
- The two alternative depths for the same discharge and energy have been called *alternate* depths.

That terminology seems to be not quite right – alternate means "occur or cause to occur by turns, go repeatedly from one to another". *Alternative* seems better - "available as another choice", and we will use that.

- In the vicinity of the critical point, where it is easier for flow to pass from one state to another, the flow can very easily form waves (and our hydrostatic approximation would break down).
- Flows can pass from one state to the other. Consider the flow past a sluice gate in a channel as shown in Figure 1-1(c). The relatively deep slow flow passes under the gate, suffering a large reduction in momentum due to the force exerted by the gate and emerging as a shallower faster flow, but with the same energy. These are, for example, the conditions at the points labelled 1 and 2 respectively in Figure 2-1. If we have a flow with head corresponding to that at the point 1 with surface elevation η_1 then the alternative depth is η_2 as shown. It seems that it is not possible to go in the other direction, from super-critical flow to sub-critical flow without some loss of energy, but nevertheless sometimes it is necessary to calculate the corresponding sub-critical depth. The mathematical process of solving either problem, equivalent to reading off the depths on the graph, is one of solving the equation

$$\underbrace{\frac{\alpha Q^2}{2gA^2(\eta_1)} + \eta_1}_{H_1} = \underbrace{\frac{\alpha Q^2}{2gA^2(\eta_2)} + \eta_2}_{H_2} \tag{2.2}$$

for η_2 if η_1 is given, or vice versa. Even for a rectangular section this equation is a nonlinear transcendental equation which has to be solved numerically by procedures such as Newton's method.

2.2 Critical flow

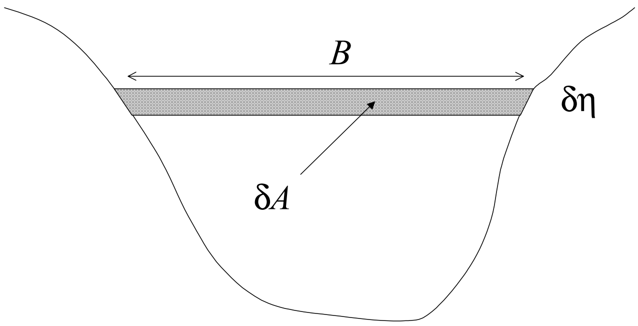

Figure 2-2. Cross-section of waterway with increment of water level

We now need to find what the condition for critical flow is, where the head is a minimum. Equation (2.1) is

$$H = \eta + \frac{\alpha}{2g}\frac{Q^2}{A^2(\eta)},$$

and critical flow is when $dH/d\eta = 0$:

$$\frac{dH}{d\eta} = 1 - \frac{\alpha Q^2}{gA^3(\eta)} \times \frac{dA}{d\eta} = 0.$$

The problem now is to evaluate the derivative $dA/d\eta$. From Figure 2-2, in the limit as $\delta\eta \to 0$ the element of area $\delta A = B\,\delta\eta$,such that $dA/d\eta = B$, the width of the free surface. Substituting, we have *the condition for critical flow*:

$$\alpha\frac{Q^2 B}{gA^3} = 1. \tag{2.3}$$

This can be rewritten as

$$\alpha \frac{(Q/A)^2}{g\,(A/B)} = 1,$$

and as $Q/A = U$, the mean velocity over the section, and $A/B = D$, the *mean* depth of flow, this means that

$$\text{Critical flow occurs when} \quad \alpha \frac{U^2}{gD} = 1, \quad \text{that is, when} \quad \alpha \times \frac{(\text{Mean velocity})^2}{g \times \text{Mean depth}} = 1. \tag{2.4}$$

We write this as

$$\alpha F^2 = 1 \quad \text{or} \quad \sqrt{\alpha} F = 1, \tag{2.5}$$

where the symbol F is the *Froude number*, defined by:

$$F = \frac{Q/A}{\sqrt{gA/B}} = \frac{U}{\sqrt{gD}} = \frac{\text{Mean velocity}}{\sqrt{g \times \text{Mean depth}}}.$$

The usual statement in textbooks is that "critical flow occurs when the Froude number is 1". We have chosen to generalise this slightly by allowing for the coefficient α not necessarily being equal to 1, giving $\alpha F^2 = 1$ at critical flow. Any form of the condition, equation (2.3), (2.4) or (2.5) can be used. The mean depth at which flow is critical is the "critical depth":

$$D_c = \alpha \frac{U^2}{g} = \alpha \frac{Q^2}{gA^2}. \tag{2.6}$$

2.3 The Froude number

The dimensionless Froude number is traditionally used in hydraulic engineering to express the relative importance of inertia and gravity forces, and occurs throughout open channel hydraulics. It is relevant where the water has a free surface. It almost always appears in the form of αF^2 rather than F. It might be helpful here to define F by writing

$$F^2 = \frac{Q^2 B}{gA^3}.$$

Consider a calculation where we attempt to quantify the relative importance of kinetic and potential energies of a flow – and as the depth is the only vertical scale we have we will use that to express the potential energy. We write

$$\frac{\text{Mean kinetic energy per unit mass}}{\text{Mean potential energy per unit mass}} = \frac{\frac{1}{2}\alpha U^2}{gD} = \tfrac{1}{2}\alpha F^2,$$

which indicates something of the nature of the dimensionless number αF^2.

Flows which are fast and shallow have large Froude numbers, and those which are slow and deep have small Froude numbers. For example, consider a river or canal which is $2\,\mathrm{m}$ deep flowing at $0.5\,\mathrm{m\,s^{-1}}$ (make some effort to imagine it - we can well believe that it would be able to flow with little surface disturbance!). We have

$$F = \frac{U}{\sqrt{gD}} \approx \frac{0.5}{\sqrt{10 \times 2}} = 0.11 \quad \text{and} \quad F^2 = 0.012\,,$$

and we can imagine that the rough relative importance of the kinetic energy contribution to the potential contribution really might be of the order of this 1%. Now consider flow in a street gutter after rain. The velocity might also be $0.5\,\mathrm{m\,s^{-1}}$, while the depth might be as little as $2\,\mathrm{cm}$. The Froude number is

$$F = \frac{U}{\sqrt{gD}} \approx \frac{0.5}{\sqrt{10 \times 0.02}} = 1.1 \quad \text{and} \quad F^2 = 1.2\,,$$

which is just super-critical, and we can easily imagine it to have many waves and disturbances on it due to irregularities in the gutter.

It is clear that αF^2 expresses the scale of the importance of kinetic energy to potential energy, even if not in a $1 : 1$ manner (the factor of $1/2$). It seems that αF^2 is a better expression of the relative importance than the traditional use of F. In fact, we suspect that as it always seems to appear in the form $\alpha F^2 = \alpha U^2/gD$, we could define an improved Froude number, $F_{\text{improved}} = \alpha U^2/gD$, which explicitly recognises (a) that U^2/gD is more fundamental than $U/\sqrt{gD}$, and (b) that it is the **weighted** value of u^2 over the whole section, αU^2, which better expresses the importance of dynamic contributions. However, we will use the traditional definition $F = U/\sqrt{gD}$. In tutorials, assignments and exams, unless advised otherwise, you may assume $\alpha = 1$, as has been almost universally done in textbooks and engineering practice. However we will retain α as a parameter in these lecture notes, and we recommend it also in professional practice. Retaining it will, in general, give more accurate results, but also, retaining it while usually not being quite sure of its actual value reminds us that we should not take numerical results as accurately or as seriously as we might. Note that, in the spirit of this, we might well use $g \approx 10$ in practical calculations!

Rectangular channel

There are some special simple features of rectangular channels. These are also applicable to wide channels, where the section properties do not vary much with depth, and they can be modelled by equivalent rectangular channels, or more usually, purely in terms of a unit width. We now find the conditions for critical flow in a rectangular section of breadth b and depth h. We have $A = bh$. From equation (2.3) the condition for critical flow for this section is:

$$\frac{\alpha Q^2}{gb^2h^3} = 1, \tag{2.7}$$

but as $Q = Ubh$, this is the condition

$$\frac{\alpha U^2}{gh} = 1. \tag{2.8}$$

Some useful results follow if we consider the *volume flow per unit width* q:

$$q = \frac{Q}{b} = \frac{Ubh}{b} = Uh. \tag{2.9}$$

Eliminating Q from (2.7) or U from (2.8) or simply using (2.6) with $D_c = h_c$ for the rectangular section gives the critical depth, when H is a minimum:

$$h_c = \left(\alpha \frac{q^2}{g}\right)^{1/3}. \tag{2.10}$$

This shows that the critical depth h_c for rectangular or wide channels depends only on the flow per unit width, and not on any other section properties. As for a rectangular channel it is obvious and convenient to place the origin on the bed, such that $\eta = h$. Then equation (2.1) for critical conditions when H is a minimum, $H = H_c$ becomes

$$H_c = h_c + \frac{\alpha}{2g}\frac{Q^2}{A_c^2} = h_c + \frac{\alpha}{2g}\frac{Q^2}{b^2h_c^2} = h_c + \frac{\alpha q^2}{2g}\frac{1}{h_c^2},$$

and using equation (2.10) to eliminate the q^2 term:

$$H_c = h_c + \frac{h_c^3}{2}\frac{1}{h_c^2} = \frac{3}{2}h_c \quad \text{or,} \quad h_c = \frac{2}{3}H_c. \tag{2.11}$$

2.4 Water level changes at local transitions in channels

Now we consider some simple transitions in open channels from one bed condition to another.

Sub-critical flow over a step in a channel or a narrowing of the channel section: Consider the

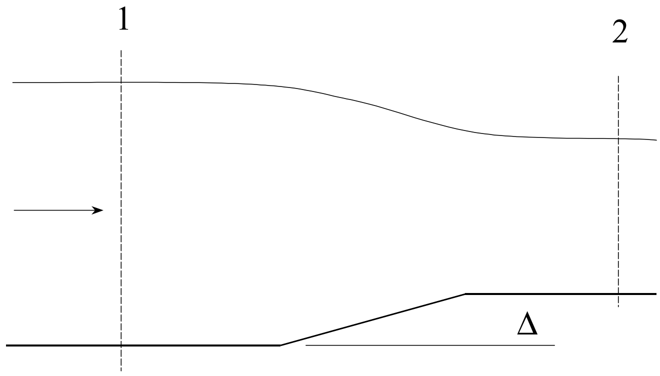

Figure 2-3. Subcritical flow passing over a rise in the bed

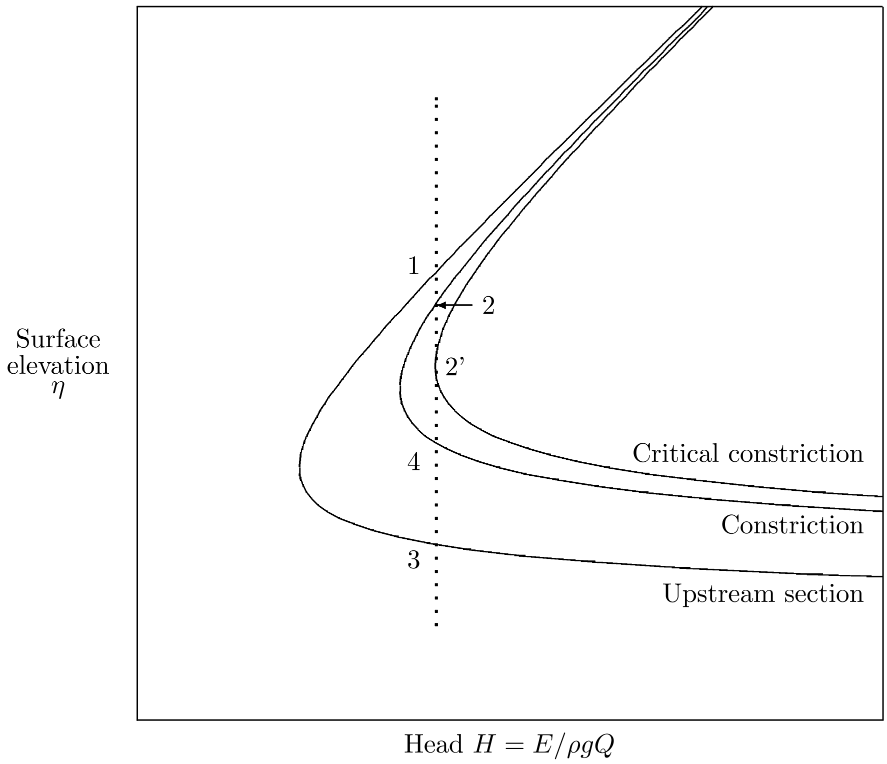

Figure 2-4. Head/Surface-elevation relationships for three cross-sections

flow as shown in Figure 2-3. At the upstream section the (H, η) diagram can be drawn as indicated in Figure 2-4. Now consider another section at an elevation and possible constriction of the channel. The corresponding curve on Figure 2-4 goes to infinity at the higher value of $z_{\min}$ and the curve can be shown to be pushed to the right by this raising of the bed and/or a narrowing of the section. At this stage it is not obvious that the water surface does drop down as shown in Figure 2-3, but it is immediately explained if we consider the point 1 on Figure 2-4 corresponding to the initial conditions. As we assume that no energy is lost in travelling over the channel constriction, the surface level must be as shown at point 2 on Figure 2-4, directly below 1 with the same value of H, and we see how, possibly against expectation, the surface really must drop down if subcritical flow passes through a constriction.

Sub-critical flow over a step or a narrowing of the channel section causing critical flow: Consider now the case where the step Δ is high enough and/or the constriction narrow enough that the previously sub-critical flow is brought to critical, going from point 1 as before, but this time going to point 2' on Figure 2-4. This shows that for the given discharge, the section cannot be constricted more than this amount which would just take it to critical. Otherwise, the (H, η) curve for this section would be moved further to the right and there would be no real depth solutions and no flow possible. In this case the flow in the constriction would remain critical but the upstream depth would have to increase so as to make the flow possible. The step is then acting as a weir, controlling the flow such that there is a unique relationship between flow and depth.

Super-critical flow over a step in a channel or a narrowing of the channel section: Now consider super-critical flow over the same constriction as shown in Figure 2-5. In this case the depth actually increases as the water passes over the step, going from 3 to 4, as the construction in Figure 2-4 shows.

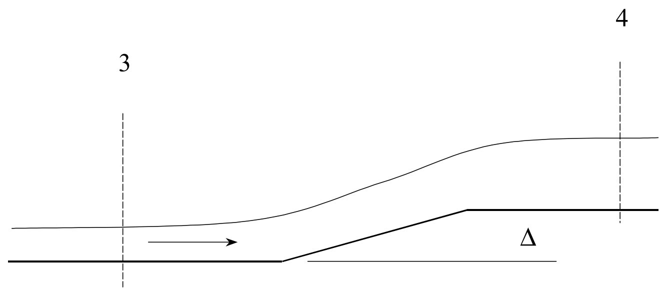

Figure 2-5. Supercritical flow passing over a hump in the bed.

The mathematical problem in each of these cases is to solve an equation similar to (2.2) for η_2, expressing the fact that the head is the same at the two sections:

$$\underbrace{\frac{\alpha Q^2}{2gA_1^2(\eta_1)} + \eta_1}_{H_1} = \underbrace{\frac{\alpha Q^2}{2gA_2^2(\eta_2)} + \eta_2}_{H_2} . \tag{2.12}$$

As the relationship between area and elevation at 2 is different from that at 1, we have shown two different functions for area as a function of elevation, $A_1(\eta_1)$ and $A_2(\eta_2)$.

Example: A rectangular channel of width b_1 carries a flow of Q, with a depth h_1. The channel section is narrowed to a width b_2 and the bed raised by Δ, such that the flow depth above the bed is now h_2. Set up the equation which must be solved for h_2.
Equation (2.12) can be used. If we place the datum on the bed at 1, then $\eta_1 = h_1$ and $A_1(\eta_1) = b_1\eta_1 = b_1h_1$. Also, $\eta_2 = \Delta + h_2$ and $A_2(\eta_2) = b_2(\eta_2 - \Delta) = b_2h_2$. The equation becomes

$$\begin{aligned} \frac{\alpha Q^2}{2gb_1^2h_1^2} + h_1 &= \frac{\alpha Q^2}{2gb_2^2h_2^2} + \Delta + h_2, \quad \textit{to be solved for } h_2\textit{, OR,} \\ \frac{\alpha Q^2}{2gb_1^2h_1^2} + h_1 &= \frac{\alpha Q^2}{2gb_2^2(\eta_2 - \Delta)^2} + \eta_2, \quad \textit{to be solved for } \eta_2. \end{aligned}$$

In either case the equation, after multiplying through by h_2 or η_2 respectively, becomes a cubic, which has no simple analytical solution and generally has to be solved numerically. Below we will present methods for this.

2.5 Some practical considerations

2.5.1 Trapezoidal sections

Most canals are excavated to a trapezoidal section, and this is often used as a convenient approximation

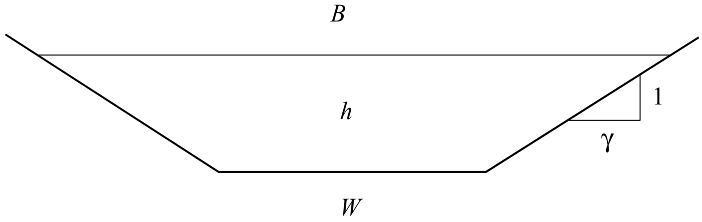

Figure 2-6. Trapezoidal section showing important quantities

to river cross-sections too. In many of the problems in this course we will consider the case of trapezoidal sections. We will introduce the terms defined in Figure 2-6: the bottom width is W, the depth is h, the top width is B, and the *batter slope,* defined to be the ratio of H:V dimensions is γ. From these the following important section properties are easily obtained:

$$\begin{aligned} \text{Top width} &: \quad B = W + 2\gamma h \\ \text{Area} &: \quad A = h\,(W + \gamma h) \\ \text{Wetted perimeter} &: \quad P = W + 2\sqrt{1+\gamma^2}h, \end{aligned}$$

where we will see that the wetted perimeter is an important quantity when we consider friction in channels. (*Ex. Obtain these relations*).

2.5.2 Solution methods for alternative depths

Here we consider the problem of solving equation (2.12) numerically:

$$\frac{\alpha Q^2}{2gA_1^2(\eta_1)} + \eta_1 = \frac{\alpha Q^2}{2gA_2^2(\eta_2)} + \eta_2,$$

where we assume that we know the upstream conditions at point 1 and we have to find η_2. The right side shows sufficiently complicated dependence on η_2 that even for rectangular sections we have to solve this problem numerically. Reference can be made to any book on numerical methods for solving nonlinear equations, but here we briefly describe some techniques and then develop a simplified version of a robust method

1. Trial and error - evaluate the right side of the equation with various values of η_2 until it agrees with the left side. This is simple, but slow to converge and not suitable for machine computation.
2. Direct iteration - re-arrange the equation in the form

$$\eta_2 = H_1 - \frac{\alpha Q^2}{2gA_2^2(\eta_2)}$$

   and successively evaluate the right side and substitute for η_2. We can show that this converges only if the flow at 2 is subcritical ($\alpha F^2 < 1$), the more common case. Provided one is aware of that limitation, the method is simple to apply.
3. Bisection - choose an initial interval in which it is known a solution lies (the value of the function changes sign), then successively halve the interval and determine in which half the solution lies each time until the interval is small enough. Robust, not quite as simply programmed, but will always converge to a solution.
4. Newton's method - make an estimate and then make successively better ones by travelling down the local tangent. This is fast, and reliable if a solution exists. We write the equation to be solved as

$$f(\eta_2) = \eta_2 + \frac{\alpha Q^2}{2gA_2^2(\eta_2)} - H_1 \ (= 0 \text{ when the solution } \eta_2 \text{ is found}). \qquad (2.13)$$

Then, if $\eta_2^{(n)}$ is the nth estimate of the solution, Newton's method gives a better estimate:

$$\eta_2^{(n+1)} = \eta_2^{(n)} - \frac{f(\eta_2^{(n)})}{f'(\eta_2^{(n)})}, \tag{2.14}$$

where $f'(h_2) = \partial f / \partial \eta_2$. In our case, from (2.13):

$$f'(\eta_2) = \frac{\partial f(\eta_2)}{\partial \eta_2} = 1 - \frac{\alpha Q^2}{g A_2^3(\eta_2)} \frac{\partial A_2}{\partial \eta_2} = 1 - \frac{\alpha Q^2 B_2(\eta_2)}{g A_2^3(\eta_2)} = 1 - \alpha F^2(\eta_2),$$

which is a simple result - obtained using the procedure we used for finding critical flow in an arbitrary section. Hence, the procedure (2.14) is

$$\eta_2^{(n+1)} = \eta_2^{(n)} - \frac{\eta_2^{(n)} + \frac{\alpha Q^2}{2g A_2^2(\eta_2^{(n)})} - H_1}{1 - \alpha F_2^{(n)2}}. \tag{2.15}$$

Note that this will not converge as quickly if the flow at 2 is critical, where both numerator and denominator go to zero as the solution is approached, but the quotient is still finite. This expression looks complicated, but it is simple to implement on a computer, although is too complicated to appear on an examination paper in this course.

These methods will be examined in tutorials.

2.6 Critical flow as a control - broad-crested weirs

For a given discharge, the (H, η) diagram showed that the bed cannot be raised or the section narrowed more than the amount which would just take it to critical. Otherwise there would be no real depth solutions and no flow possible. If the channel were constricted even more, then the depth of flow over the raised bed would remain constant at the critical depth, and the upstream depth would have to increase so as to make the flow possible. The step is then acting as a weir, controlling the flow.

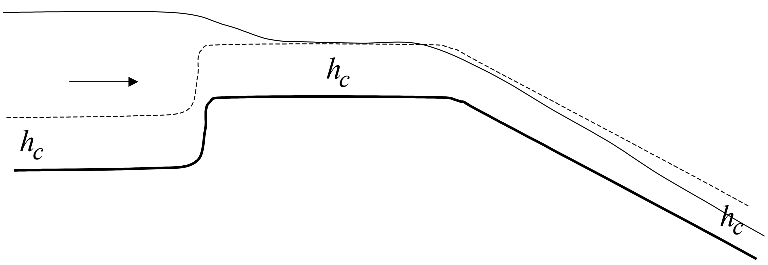

Figure 2-7. A broad-crested weir spillway, showing the critical depth over it providing a control.

Consider the situation shown in Figure 2-7 where the bed falls away after the horizontal section, such as on a spillway. The flow upstream is subcritical, but the flow downstream is fast (supercritical). Somewhere between the two, the flow depth *must* become critical - the flow reaches its critical depth at some point on top of the weir, and the weir provides a *control* for the flow, such that *a relationship between flow and depth exists*. In this case, the head upstream (the height of the upstream water surface above the sill) uniquely determines the discharge, and it is enough to measure the upstream surface elevation where the flow is slow and the kinetic part of the head negligible to provide a point on a unique relationship between that head over the weir and the discharge. No other surface elevation need be measured.

Figure 2-8 shows a horizontal flow control, a broad-crested weir, in a channel. In recent years there has been a widespread development (but not in Australia, unusually) of such broad-crested weirs placed in streams where the flow is subcritical both before and after the weir, but passes through critical on the weir. There is a small energy loss after the flume. The advantage is that it is only necessary to measure the upstream head over the weir.

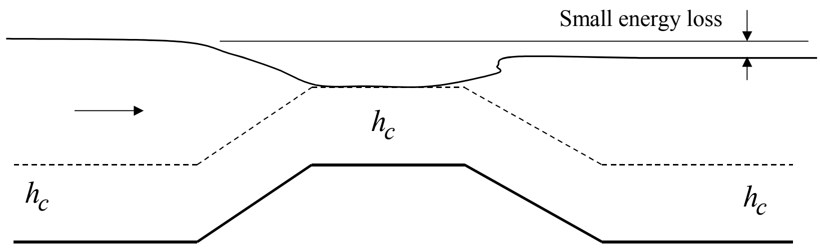

Figure 2-8. A broad-crested weir

3. Conservation of momentum in open channel flow

3.1 Integral momentum theorem

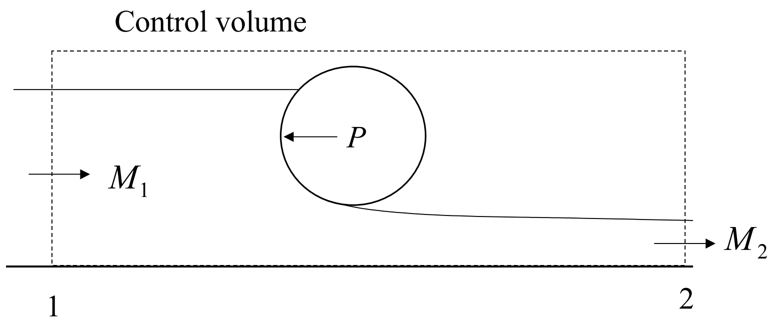

Figure 3-1. Obstacle in stream reducing the momentum flux

We have applied energy conservation principles. Now we will apply momentum. We will consider, like several problems above, relatively short reaches and channels of prismatic (constant) cross-section such that the small contributions due to friction and the component of gravity down the channel are roughly in balance. Figure 3-1 shows the important *horizontal* contributions to force and momentum in the channel, where there is a structure applying a force P to the fluid in the control volume we have drawn.

The momentum theorem applied to the control volume shown can be stated: *the net momentum flux* ***leaving*** *the control volume is equal to the net force applied* ***to the fluid*** *in the control volume*. The momentum flux is defined to be the surface integral over the control surface CS:

$$\int_{\mathrm{CS}} \left(p\,\hat{\mathbf{n}} + \mathbf{u}\,\rho\mathbf{u}.\hat{\mathbf{n}}\right)\, dS,$$

where $\hat{\mathbf{n}}$ is a unit vector normal to the surface, such that the pressure contribution on an element of area dS is the force $p\,dS$ times the unit normal vector $\hat{\mathbf{n}}$ giving its direction; $\mathbf{u}$ is the velocity vector such that $\mathbf{u}.\hat{\mathbf{n}}$ is the component of velocity normal to the surface, $\mathbf{u}.\hat{\mathbf{n}}\,dS$ is the *volume* rate of flow across the surface, multiplying by density gives the *mass* rate of flow across the surface $\rho\mathbf{u}.\hat{\mathbf{n}}\,dS$, and multiplying by *velocity* gives $\mathbf{u}\,\rho\mathbf{u}.\hat{\mathbf{n}}\,dS$, the momentum rate of flow across the surface.

We introduce $\mathbf{i}$, a unit vector in the x direction. On the face 1 of the control surface in Figure 3-1, as the outwards normal is in the upstream direction, we have $\hat{\mathbf{n}} = -\mathbf{i}$, and $\mathbf{u} = u_1\mathbf{i}$, giving $\mathbf{u}.\hat{\mathbf{n}} = -u_1$and the vector momentum flux across face 1 is

$$\mathbf{M}_1 = -\mathbf{i} \int_{A_1} \left(p_1 + \rho u_1^2\right)\, dA = -\mathbf{i}\, M_1,$$

where the scalar quantity

$$M_1 = \int_{A_1} \left(p_1 + \rho u_1^2\right)\, dA.$$

Similarly, on face 2 of the control surface, as the outwards normal is in the downstream direction, we have $\hat{\mathbf{n}} = \mathbf{i}$ and $\mathbf{u} = u_2\mathbf{i}$, giving $\mathbf{u}.\hat{\mathbf{n}} = +u_2$ and the vector momentum flux across face 2 is

$$\mathbf{M}_2 = +\mathbf{i} \int_{A_2} \left(p_2 + \rho u_2^2\right)\, dA = +\mathbf{i}\, M_2$$

with scalar quantity

$$M_2 = \int_{A_2} \left(p_2 + \rho u_2^2\right)\, dA.$$

Using the momentum theorem, and recognising that the horizontal component of the force of the body *on the fluid* is $-P\,\mathbf{i}$, then we have, writing it as a vector equation but including only x ($\mathbf{i}$) components:

$$\mathbf{M}_1 + \mathbf{M}_2 = -P\,\mathbf{i}$$

As $\mathbf{M}_1 = -M_1\mathbf{i}$ and $\mathbf{M}_2 = +M_2\mathbf{i}$, we can write it as a scalar equation giving:

$$P = M_1 - M_2, \tag{3.1}$$

where P is the force of the water on the body (or bodies).

3.1.1 Momentum flux across a section of channel

From the above, it can be seen how useful is the concept of the horizontal *momentum flux* at a section of the flow in a waterway:

$$M = \int_A \left(p + \rho u^2\right)\, dA.$$

We attach different signs to the contributions depending on whether the fluid is leaving (+ve) or entering (-ve) the control volume. As elsewhere in these lectures on open channel hydraulics we use the hydrostatic approximation for the pressure: $p = \rho g(\eta - z)$, which gives

$$M = \rho \int_A \left(g(\eta - z) + u^2\right)\, dA.$$

Now we evaluate this in terms of the quantities at the section.

Pressure and elevation contribution $\rho \int_A g(\eta - z)\, dA$: The integral $\int_A (\eta - z)\, dA$ is simply the first moment of area about a transverse horizontal axis at the surface, we can write it as

$$\int_A (\eta - z)\, dA = A\bar{h}, \tag{3.2}$$

where $\bar{h}$ is the *depth* of the centroid of the section *below the surface*.

Velocity contribution $\rho \int_A u^2\, dA$: Now we have the task of evaluating the square of the horizontal velocity over the section. As with the kinetic energy integral, it seems that the sensible thing to do is to recognise that all velocity components and velocity fluctuations will be of a scale given by the mean flow velocity in the stream at that point, and so we simply write

$$\int_A u^2\, dA = \beta U^2 A = \beta \frac{Q^2}{A}, \tag{3.3}$$

which defines β as a coefficient which will be somewhat greater than unity, given by

$$\beta = \frac{\int_A u^2\, dA}{U^2 A}. \tag{3.4}$$

This coefficient is known as a *Boussinesq* coefficient, in honour of the French engineer who introduced it, who did much important work in the area of the non-uniformity of velocity and the non-hydrostatic nature of the pressure distribution. Most presentations of open channel theory adopt the approximation that there is no variation of velocity over the section, such that it is assumed that $\beta = 1$. Typical real values are $\beta = 1.05 - 1.15$, somewhat less than the Coriolis energy coefficient α.

Combining: We can substitute to give the expression we will use for the Momentum Flux:

$$M = \rho\left(gA\bar{h} + \beta U^2 A\right) = \rho\left(gA\bar{h} + \beta\frac{Q^2}{A}\right) = \rho g\left(A(h)\,\bar{h}(h) + \frac{\beta Q^2}{g}\frac{1}{A(h)}\right) \tag{3.5}$$

where we have shown the dependence on depth in each term. This expression can be compared with that for the head as defined in equation (2.1) but here expressed relative to the bottom of the channel:

$$H = h + \frac{\alpha Q^2}{2g}\frac{1}{A^2(h)}.$$

The variation with h is different between this and equation (3.5). For large h, $H \sim h$, while $M \sim A(h)\,\bar{h}(h)$, which for a rectangular section goes like h^2. For small h, $H \sim 1/A^2(h)$, and $M \sim 1/A(h)$.

Note that we can re-write equation (3.5) in terms of Froude number (actually appearing as F^2 – *yet again*) to indicate the relative importance of the two parts, which we could think of as "static" and "dynamic" contributions:

$$M = \rho g A\bar{h}\left(1+\beta F^2\frac{A/B}{\bar{h}}\right).$$

The ratio $(A/B)/\bar{h}$, mean depth to centroid depth, will have a value typically of about 2.

Example: *Calculate (a) Head (using the channel bottom as datum) and (b) Momentum flux, for a rectangular section of breadth* b *and depth* h.

We have $A = bh$, $\bar{h} = h/2$. Substituting into equations (2.1) and (3.5) we obtain

$$\begin{aligned} H &= h + \frac{\alpha Q^2}{2gb^2} \times \frac{1}{h^2} \quad \text{and,} \\ M &= \rho\left(\frac{gb}{2} \times h^2 + \frac{\beta Q^2}{b} \times \frac{1}{h}\right). \end{aligned}$$

Note the quite different variation with h between the two quantities.

3.1.2 Minimum momentum flux and critical depth

We calculate the condition for minimum M:

$$\frac{\partial M}{\partial h} = \frac{\partial}{\partial h}(A(h)\bar{h}(h)) - \frac{\beta Q^2}{g}\frac{1}{A^2(h)}\frac{\partial A}{\partial h} = 0. \tag{3.6}$$

The derivative of the first moment of area about the surface is obtained by considering the surface increased by an amount $h + \delta h$

$$\frac{\partial(A\bar{h})}{\partial h} = \lim_{\delta h \to 0}\frac{(A(h)\bar{h}(h))_{h+\delta h} - A(h)\bar{h}(h))}{\delta h}. \tag{3.7}$$

The situation is as shown in Figure 3-2. The first moment of area about an axis transverse to the channel

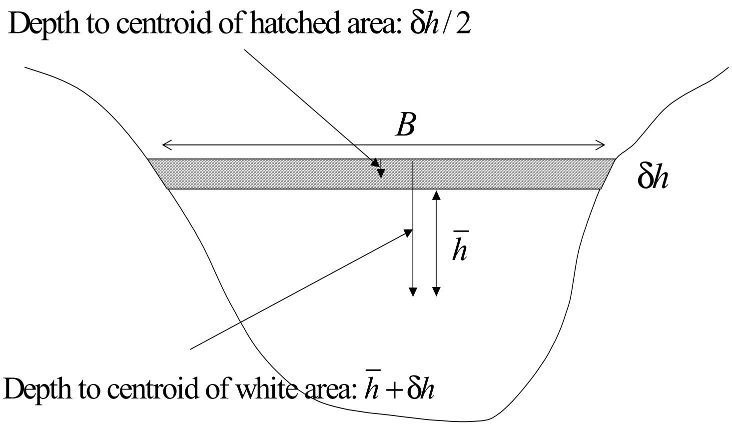

Figure 3-2. Geometrical interpretation of calculation of position of centroid

at the new surface is:

$$(A(h)\bar{h}(h))_{h+\delta h} = A(h) \times (\bar{h} + \delta h) + B \times \delta h \times \frac{\delta h}{2},$$

so that, substituting into equation (3.7), in the limit $\delta h \to 0$,

$$\frac{\partial (A\bar{h})}{\partial h} = \lim_{\delta h \to 0} \frac{A(h) \times (\bar{h} + \delta h) + B \times \delta h \times \delta h/2 - A(h)\bar{h}(h))}{\delta h} = A(h) = A, \qquad (3.8)$$

which is surprisingly simple. Substituting both this and $\partial A/\partial h = B$ in equation (3.6), we get the condition for minimum M:

$$\frac{\beta Q^2 B}{g A^3} = \beta F^2 = 1, \qquad (3.9)$$

which is a similar condition for the minimum energy, but as in general $\alpha \neq \beta$, the condition for minimum momentum is not the same as that for minimum energy.

3.1.3 Momentum flux -depth diagram

If the cross-section changes or there are other obstacles to the flow, the sides of the channel and/or the obstacles will also exert a force along the channel on the fluid. We can solve for the total force exerted between two sections if we know the depth at each. In the same way as we could draw an (H, η) diagram for a given channel section, we can draw an (M, η) diagram. It is more convenient here to choose the datum on the bed of the channel so that we can interpret the surface elevation η as the depth h. Figure 3-3 shows a momentum flux – depth (M, h) diagram. Note that it shows some of the main features of the (H, h) diagram, with two possible depths for the same momentum flux – called *conjugate* depths. However the limiting behaviours for small and large depths are different for momentum, compared with energy.

3.2 Flow under a sluice gate and the hydraulic jump

Consider the flow problem shown at the top of Figure 3-4, with sub-critical flow (section 0) controlled by a sluice gate. The flow emerges from under the gate flowing fast (super-critically, section 1). There has been little energy loss in the short interval 0-1, but the force of the gate on the flow has substantially reduced its momentum flux. It could remain in this state, however here we suppose that the downstream level is high enough such that a hydraulic jump occurs, where there is a violent turbulent motion and in a short distance the water changes to sub-critical flow again. In the jump there has been little momentum loss, but the turbulence has caused a significant loss of energy between 1-2. After the jump, at stage 2, the flow is sub-critical again. We refer to this depth as being *sequent* to the original depth.

In the bottom part of Figure 3-4 we combine the (H, h) and (M, h) diagrams, so that the vertical axis is

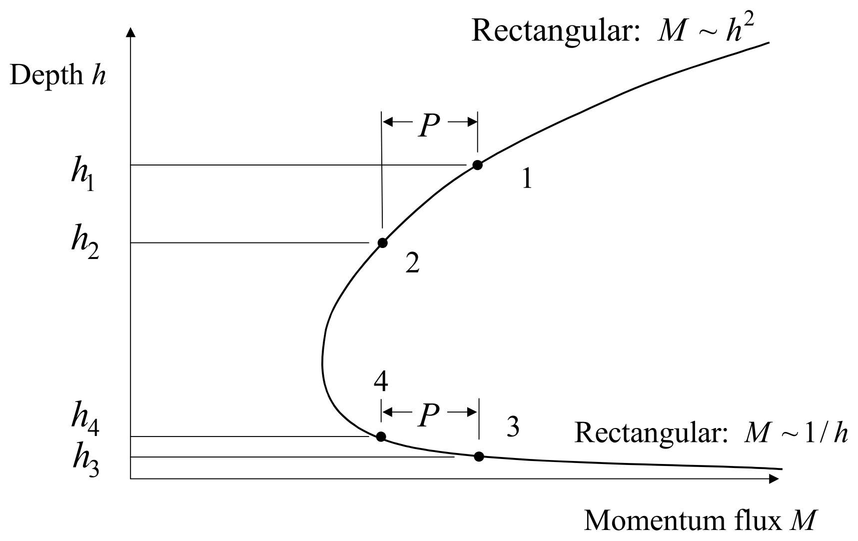

Figure 3-3. Momentum flux – depth diagram, showing effects of a momentum loss P for subcritical and supercritical flow.

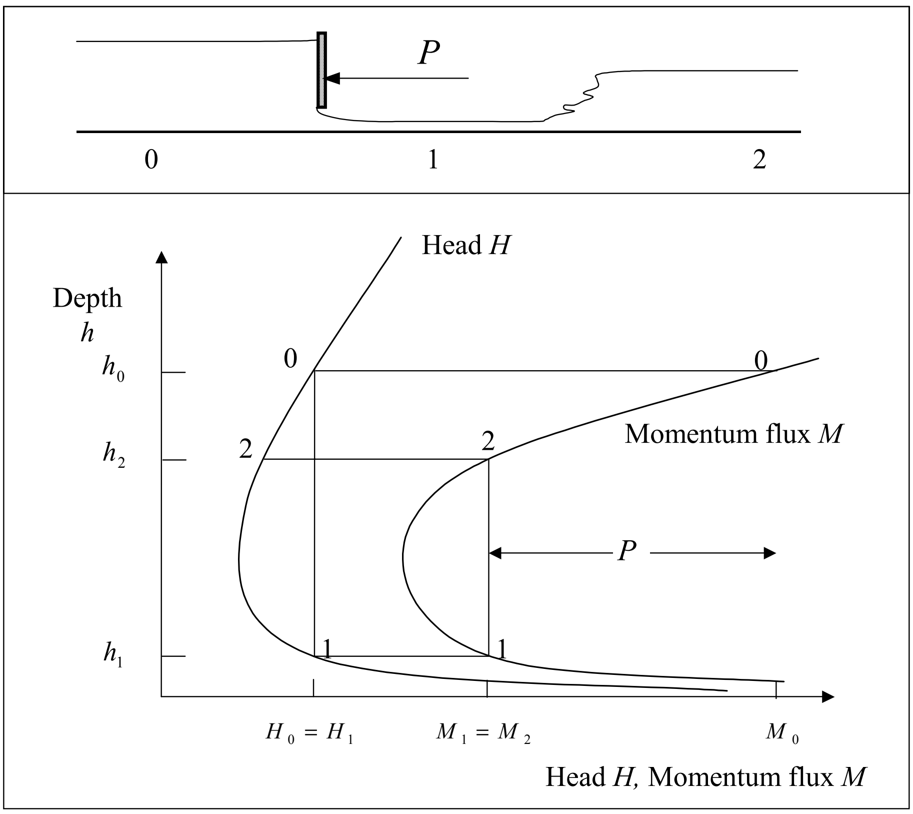

Figure 3-4. Combined Head and Momentum diagrams for the sluice gate and hydraulic jump problem

depth h and the two horizontal axes are head H and momentum flux M, with different scales. We now outline the procedure we follow to analyse the problem of flow under a sluice gate, with upstream force P, and a subsequent hydraulic jump.

- We are given the discharge Q and the upstream depth h_0, and we know the cross-sectional details of the channel.
- We can compute the energy and momentum at 0, H_0 and M_0 (see points 0 on the $M-H-h$ plot).
- As energy is conserved between 0 and 1, the depth h_1 can be calculated by solving the energy equation with $H_1 = H_0$, possibly using Newton's method.
- In fact, this depth may not always be realisable, if the gate is not set at about the right position. The flow at the lip of the gate leaves it vertically, and turns around to horizontal, so that the gate opening must be larger than h_1. A rough guide is that the gate opening must be such that $h_1 \approx 0.6\times$ Gate opening.
- With this h_1 we can calculate the momentum flux M_1.
- The force on the gate P (assuming that the channel is prismatic) can be calculated from:

$$P = M_0 - M_1 = \rho\left(gA\overline{h}+\beta\frac{Q^2}{A}\right)_0 - \rho\left(gA\overline{h}+\beta\frac{Q^2}{A}\right)_1$$

- Across the hydraulic jump momentum is conserved, such that $M_2 = M_1$:

$$\left(gA\overline{h}+\beta\frac{Q^2}{A}\right)_2 = \left(gA\overline{h}+\beta\frac{Q^2}{A}\right)_1$$

- This gives a nonlinear equation for h_2 to be solved numerically (note that A and $\bar{h}$ are both functions of h). In the case of a *rectangular channel* the equation can be written

$$\frac{1}{2}gh_1^2 + \frac{\beta q^2}{h_1} = \frac{1}{2}gh_2^2 + \frac{\beta q^2}{h_2},$$

where $q = Q/b$, the discharge per unit width. In fact it can be solved analytically. Grouping like terms on each side and factorising:

$$(h_2 - h_1)(h_2 + h_1) = \frac{2\beta q^2}{g}\left(\frac{1}{h_1}-\frac{1}{h_2}\right),$$

$$h_2^2\, h_1 + h_1^2 h_2 - \frac{2\beta q^2}{g} = 0,$$

which is a quadratic in h_2, with solutions

$$h_2 = -\frac{h_1}{2} \pm \sqrt{\frac{h_1^2}{4} + \frac{2\beta q^2}{gh_1}},$$

but we cannot have a negative depth, and so only the positive sign is taken. Dividing through by h_1:

$$\frac{h_2}{h_1} = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\beta q^2}{gh_1^3}} = \frac{1}{2}\left(\sqrt{1+8\beta F_1^2}-1\right)$$

- Sometimes the actual depth of the downstream flow is determined by the boundary condition further downstream. If it is not deep enough the actual jump may be an *undular hydraulic jump*, which does not dissipate as much energy, with periodic waves downstream.
- The pair of depths (h_1, h_2) for which the flow has the same momentum are traditionally called the *conjugate* depths.

- The loss in energy $H_2 - H_1$ can be calculated. For a rectangular channel it can be shown that

$$\Delta H = H_1 - H_2 = \frac{(h_2 - h_1)^3}{4 h_1 h_2}.$$

3.3 The effects of streams on obstacles and obstacles on streams

3.3.1 Interpretation of the effects of obstacles in a flow

Slow (sub-critical) approach flow Figure 3-5 shows that the effect of a drag force is to lower the water surface (counter-intuitive!?) if the flow is slow (sub-critical).

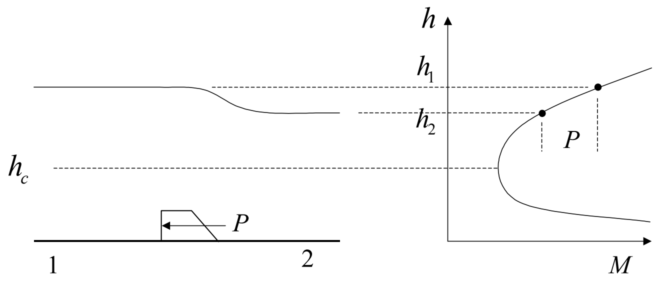

Figure 3-5. Effect of obstacles on a subcritical flow

Fast (super-critical) approach flow Figure 3-6 shows that the effect of a drag force on a super-critical flow is to raise the water surface. In fact, the effect of the local force only spreads gradually through the stream by turbulent diffusion, and the predicted change in cross-section will apply some distance downstream where the flow has become uniform (rather further than in the diagrams here).

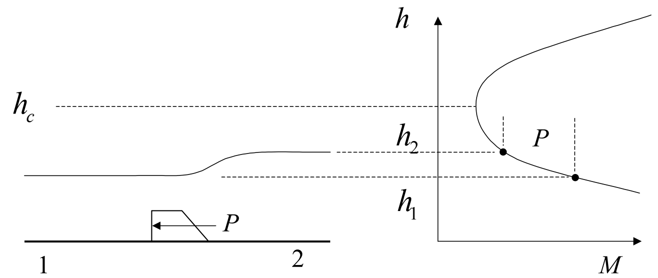

Figure 3-6. Effect of obstacles on a supercritical flow

A practical example is the fast flow downstream of a spillway, shown in Figure 3-7, where the flow becomes subcritical via a hydraulic jump. If spillway blocks are used, the water level downstream need not be as high, possibly with large savings in channel construction.

3.3.2 Bridge piers - slow approach flow

Consider flow past bridge piers as shown in Figure 3-8. As the bridge piers extend throughout the flow, for the velocity on the pier we will take the mean upstream velocity $V = Q/A_1$, and equation (3.14) can

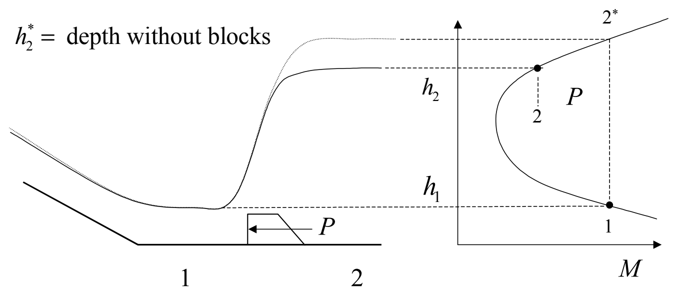

Figure 3-7. Effect of spillway blocks on lowering the water level in a spillway pool

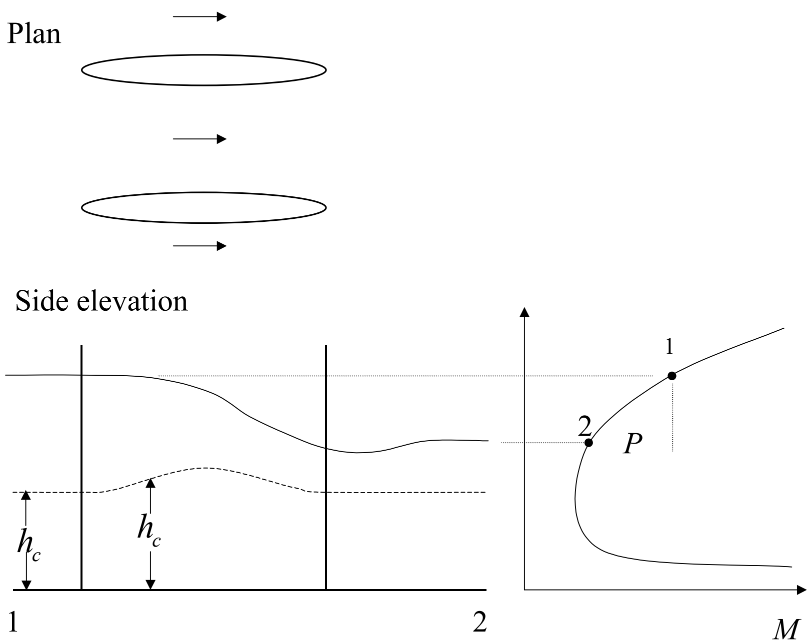

Figure 3-8. Flow past bridge piers and their effect on the flow

be used.

3.3.3 Flow in a narrowing channel - choked flow

We consider cases where the width reduction is more than in a typical bridge pier problem, such that the flow in the throat may become critical, the throat becomes a control, and the flow is said to be *choked*. If so, the upstream depth is increased, to produce a larger momentum flux there so that the imposed force due to the convergence now just produces critical flow in the throat. In problems such as these, it is very helpful to remember that *for a rectangular section*, equation (2.10):

$$h_c = \left(\alpha q^2/g\right)^{1/3}, \quad \text{or, re-written,} \quad q = \sqrt{g h_c^3/\alpha},$$

where $q = Q/b$, the flow per unit width, and also to observe that at critical depth, equation (2.11):

$$H = h_c + \frac{\alpha Q^2}{2gb^2h_c^2} = h_c + \frac{\alpha q^2}{2gh_c^2} = \frac{3}{2}h_c, \quad \text{so that} \quad h_c = \frac{2}{3}H.$$

It is clear that by reducing b, $q = Q/b$ is increased, until in this case, criticality is reached. While

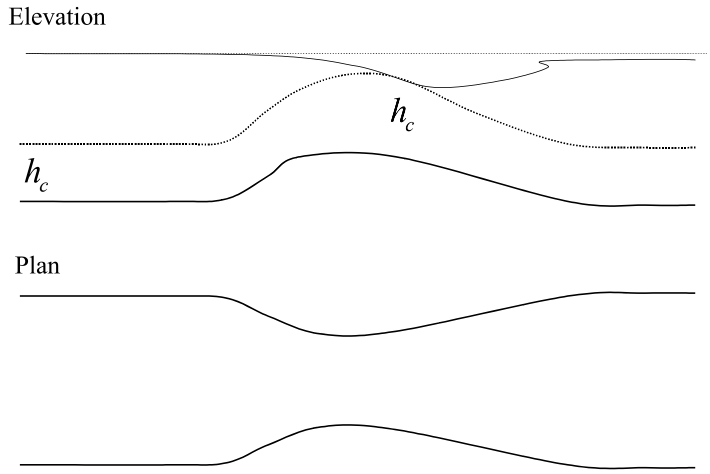

Figure 3-9. Flow through contraction sufficiently narrow that it becomes critical

generally this is not a good thing, as the bridge would then become a control, where there is a relationship between flow and depth, this becomes an advantage in flow measurement applications. In *critical flow flumes* only an upstream head is needed to calculate the flow, and the structure is deliberately designed to bring about critical depth at the throat. One way of ensuring this is by putting in a rise in the bed at the throat. Note that in the diagram the critical depth on the hump is greater than that upstream because the width has been narrowed.

3.3.4 Drag force on an obstacle

As well as sluices and weirs, many different types of obstacles can be placed in a stream, such as the piers of a bridge, blocks on the bed, Iowa vanes, the bars of a trash-rack *etc.* or possibly more importantly, the effects of trees placed in rivers ("Large Woody Debris"), used in their environmental rehabilitation. It might be important to know what the forces on the obstacles are, or in flood studies, what effects the obstacles have on the river.

Substituting equation (3.5) into equation (3.1) ($P = M_1 - M_2$) gives the expression:

$$P = \rho \left(gA\bar{h} + \beta \frac{Q^2}{A} \right)_1 - \rho \left(gA\bar{h} + \beta \frac{Q^2}{A} \right)_2 , \tag{3.10}$$

so that if we know the depth upstream and downstream of an obstacle, the force on it can be calculated. Usually, however, the calculation does not proceed in that direction, as one wants to calculate the effect of the obstacle on water levels. The effects of drag can be estimated by knowing the area of the object measured transverse to the flow, a, the drag coefficient C_d, and V, the mean fluid speed past the object:

$$P = \frac{1}{2} \rho C_d V^2 a, \tag{3.11}$$

and so, substituting into equation (3.10) gives, after dividing by density,

$$\frac{1}{2} C_d V^2 a = \left(gA\bar{h} + \beta \frac{Q^2}{A} \right)_1 - \left(gA\bar{h} + \beta \frac{Q^2}{A} \right)_2 . \tag{3.12}$$

We will write the velocity V on the obstacle as being proportional to the upstream velocity, such that we write

$$V^2 = \gamma_d \left(\frac{Q}{A_1} \right)^2 , \tag{3.13}$$

where γ_d is a coefficient which recognises that the velocity which impinges on the object is generally not equal to the mean velocity in the flow. For a small object near the bed, γ_d could be quite small; for an object near the surface it will be slightly greater than 1; for objects of a vertical scale that of the whole depth, $\gamma_d \approx 1$. Equation (3.12) becomes

$$\frac{1}{2}\gamma_d C_d \frac{Q^2}{A_1^2} a = \left(gA\bar{h} + \beta\frac{Q^2}{A}\right)_1 - \left(gA\bar{h} + \beta\frac{Q^2}{A}\right)_2 \tag{3.14}$$

A typical problem is where the downstream water level is given (sub-critical flow, so that the control is downstream), and we want to know by how much the water level will be raised upstream if an obstacle is installed. As both A_1 and $\bar{h}_1$ are functions of h_1, the solution is given by solving this transcendental equation for h_1. In the spirit of approximation which can be used in open channel hydraulics, and in the interest of simplicity and insight, we now obtain an approximate solution.

3.3.5 An approximate method for estimating the effect of channel obstructions on flooding

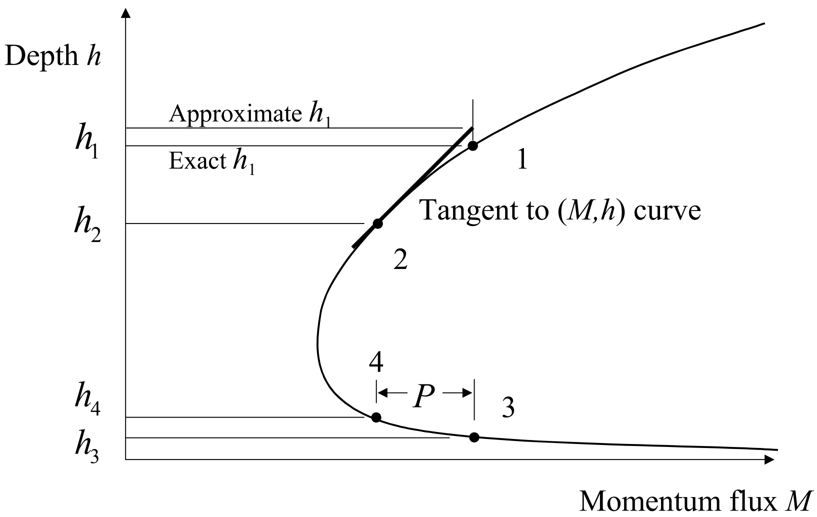

Figure 3-10. Momentum flux – depth diagram showing the approximate value of d_1 calculated by approximating the curve by its tangent at 2.

Now an approximation to equation (3.14) will be obtained which enables a direct calculation of the change in water level due to an obstacle, without solving the transcendental equation. We consider a linearised version of the equation, which means that locally we assume a straight-line approximation to the momentum diagram, for a small reduction in momentum, as shown in Figure 3-10.

Consider a small change of surface elevation δh going from section 1 to section 2, and write the expression for the downstream area

$$A_2 = A_1 + B_1 \delta h.$$

It has been shown above (equation 3.8) that

$$\frac{\partial (A\bar{h})}{\partial h} = A,$$

and so we can write an expression for $A_2\bar{h}_2$ in terms of $A_1\bar{h}_1$ and the small change in surface elevation:

$$A_2\bar{h}_2 = A_1\bar{h}_1 + \delta h \left.\frac{\partial (A\bar{h})}{\partial h}\right|_1 = A_1\bar{h}_1 + \delta h\, A_1,$$

and so equation (3.14) gives us, after dividing through by g:

$$\begin{aligned} \frac{1}{2}\gamma_d C_d \frac{Q^2}{gA_1^2} a &= -\delta h\, A_1 + \beta\frac{Q^2}{gA_1} - \beta\frac{Q^2}{g\,(A_1 + B_1\delta h)} \\ &= -\delta h\, A_1 + \beta\frac{Q^2}{gA_1}\left(1 - \left(1 + \frac{B_1}{A_1}\delta h\right)^{-1}\right). \end{aligned}$$

Now we use a power series expansion in δh to simplify the last term, neglecting terms like $(\delta h)^2$. For ε small, $(1+\varepsilon)^{-1} \approx 1 - \varepsilon$, and so

$$\frac{1}{2}\gamma_d C_d \frac{Q^2}{gA_1^2} a \approx -\delta h\, A_1 + \beta\frac{Q^2 B_1}{gA_1^2}\delta h.$$

We can now solve this to give an explicit approximation for δh:

$$\delta h \approx \frac{\frac{1}{2}\,\gamma_d\, C_d\, \frac{Q^2}{gA_1^3} a}{\beta\frac{Q^2 B_1}{gA_1^3} - 1}.$$

It is simpler to divide both sides by the mean depth A_1/B_1 to give:

$$\frac{\delta h}{A_1/B_1} = \frac{\frac{1}{2}\,\gamma_d\, C_d\, F_1^2\, \frac{a}{A_1}}{\beta F_1^2 - 1}.$$

We do not have to worry here that for subcritical flow we do not necessarily know the conditions at point 1, but instead we know them at the downstream point 2. Within our linearising approximation, we can use either the values at 1 or 2 in this expression, and so we generalise by dropping the subscripts altogether, so that we write

$$\frac{\delta h}{A/B} = \frac{\frac{1}{2}\,\gamma_d\, C_d\, F^2\, \frac{a}{A}}{\beta F^2 - 1} = \tfrac{1}{2}\,\gamma_d\, C_d\, \frac{a}{A} \times \frac{F^2}{\beta F^2 - 1}. \tag{3.15}$$

Thus we see that the relative change of depth (change of depth divided by mean depth) is directly proportional to the coefficient of drag and the fractional area of the blockage, as we might expect. The result is modified by a term which is a function of the square of the Froude number. For subcritical flow the denominator is negative, and so is δh, so that the surface drops, as we expect, and as can be seen when we solve the problem exactly using the momentum diagram. If upstream is supercritical, the surface rises. Clearly, if the flow is near critical ($\beta F_1^2 \approx 1$) the change in depth will be large (the gradient on the momentum diagram is vertical), when the theory will have limited validity.

***Example**: In a proposal for the rehabilitation of a river it is proposed to install a number of logs ("Large Woody Debris" or "Engineered Log Jam"). If a single log of diameter* $500\,\mathrm{mm}$ *and* $10\,\mathrm{m}$ *long were placed transverse to the flow, calculate the effect on river height. The stream is roughly* $100\,\mathrm{m}$ *wide, say* $10\,\mathrm{m}$ *deep in a severe flood, with a drag coefficient* $C_d \approx 1$. *The all-important velocities are a bit uncertain. We might assume a mean velocity of say* $6\,\mathrm{m\,s^{-1}}$, *and velocity on the log of* $2\,\mathrm{m\,s^{-1}}$. *Assume* $\beta = 1.1$.

We have the values

$$\begin{aligned} A &= 100 \times 10 = 1000\,\mathrm{m}^2, \quad a = 0.5 \times 10 = 5\,\mathrm{m}^2 \\ F^2 &= U^2/gD = 6^2/10/10 = 0.36, \quad \gamma_d = 2^2/6^2 \approx 0.1 \end{aligned}$$

and substituting into equation (3.15) gives

$$\frac{\delta h}{A/B} \approx \frac{\frac{1}{2}\gamma_d C_d\, \frac{a}{A_1}}{\beta - \frac{1}{F_1^2}} = \frac{\frac{1}{2} \times 0.1 \times 1 \times \frac{5}{1000}}{1.1 - \frac{1}{0.36}} = -1.5 \times 10^{-4},$$

so that multiplying by the mean depth, $\delta\eta = -1.5 \times 10^{-4} \times 10 = -1.5\,\mathrm{mm}$. The negative value is the change as we go downstream, thus we see that the flow upstream is raised by $1.5\,\mathrm{mm}$.

4. Uniform flow in prismatic channels

Uniform flow is where the depth does not change along the waterway. For this to occur the channel properties also must not change along the stream, such that the channel is prismatic, and this occurs only in constructed canals. However in rivers if we need to calculate a flow or depth, it is common to use a cross-section which is representative of the reach being considered, and to assume it constant for the application of this theory.

4.1 Features of uniform flow and relationships for uniform flow

- There are two forces in balance in steady flow:
  - The component of gravity downstream along the channel, and
  - the shear stress at the sides which offers resistance to the flow, which increases with flow velocity.
- If a channel is long and prismatic (slope and section do not change) then far from the effects of controls the two can be in balance, and if the flow is steady, the mean flow velocity and flow depth remain constant along the channel, giving *uniform flow*, at *normal depth*.

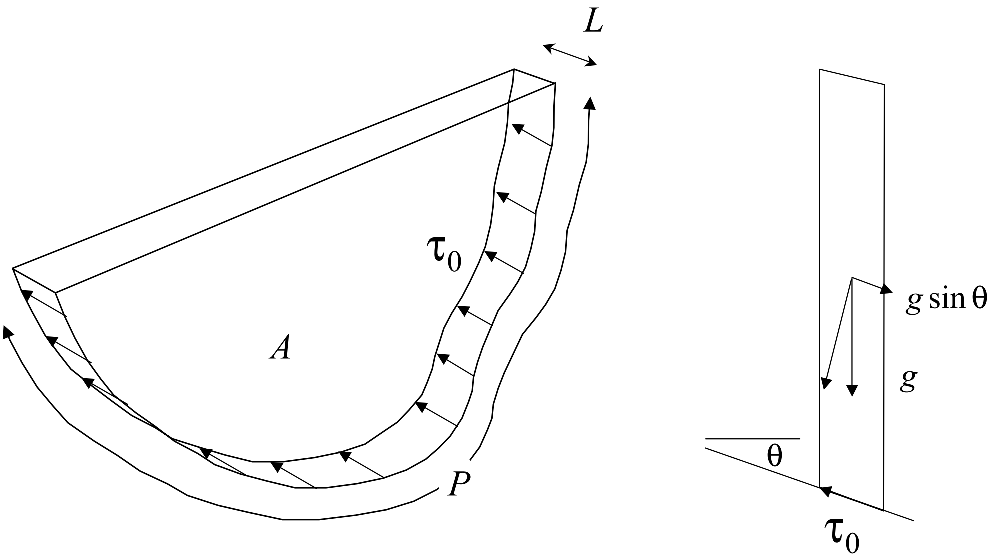

Figure 4-1. Slice of uniform channel flow showing shear forces and body forces per unit mass acting

Consider a slice of uniform flow in a channel of length L and cross-sectional area A, as shown in Figure 4-1. The component of gravity force along the channel is $\rho \times AL \times g\sin\theta$, where θ is the angle of inclination of the channel, assumed positive downwards. The shear force is $\tau_0 \times L \times P$, where τ_0 is the shear stress, and P is the wetted perimeter of the cross-section. As the two are in balance for uniform flow, we obtain

$$\frac{\tau_0}{\rho} = g\frac{A}{P}\sin\theta.$$

Now, τ_0/ρ has units of velocity squared; we combine g and the coefficient relating the mean velocity U at a section to that velocity, giving Chézy's law (1768):

$$U = C\sqrt{RS_0},$$

where C is the Chézy coefficient (with units $\mathrm{L}^{1/2}\mathrm{T}^{-1}$), $R = A/P$ is the hydraulic radius (L), and $S_0 = \sin\theta$ is the slope of the bed, positive downwards. The tradition in engineering is that we use the tangent of the slope angle, so this is valid for small slopes such that $\sin\theta \approx \tan\theta$.

- However there is experimental evidence that C depends on the hydraulic radius in the form $C \sim R^{1/6}$ (Gauckler, Manning), and the law widely used is *Manning's Law*:

$$U = \frac{1}{n} R^{2/3} S_0^{1/2},$$

where n is the Manning coefficient (units of $\mathrm{L}^{-1/3}\mathrm{T}$), which increases with increasing roughness. Typical values are: concrete - 0.013, irrigation channels - 0.025, clean natural streams - 0.03, streams with large boulders - 0.05, streams with many trees - 0.07. Usually the units are not shown.

- Multiplying by the area, Manning's formula gives the discharge:

$$Q = UA = \frac{1}{n}\frac{A^{5/3}}{P^{2/3}}\sqrt{S_0}, \tag{4.1}$$

in which both A and P are functions of the flow depth. Similarly, Chézy's law gives

$$Q = C\frac{A^{3/2}}{P^{1/2}}\sqrt{S_0}. \tag{4.2}$$

Both equations show how flow increases with cross-sectional area and slope and decreases with wetted perimeter.

4.2 Computation of normal depth

If the discharge, slope, and the appropriate roughness coefficient are known, either of equations (4.1) and (4.2) is a transcendental equation for the normal depth h_n, which can be solved by the methods described earlier. We can gain some insight and develop a simple scheme by considering a trapezoidal cross-section, where the bottom width is W, the depth is h, and the batter slopes are (H:V) $\gamma : 1$ (see Figure 2-6). The following properties are easily shown to hold (the results have already been presented above):

Top width	B	$W + 2\gamma h$
Area	A	$h\,(W + \gamma h)$
Wetted perimeter	P	$W + 2\sqrt{1+\gamma^2}h$

In the case of wide channels, (*i.e.* channels rather wider than they are deep, $h \ll W$, which is a common case) the wetted perimeter does not show a lot of variation with depth h. Similarly in the expression for the area, the second factor $W + \gamma h$ (the mean width) does not show a lot of variation with h either – most of the variation is in the first part h. Hence, if we assume that these properties hold for cross-sections of a more general nature, we can rewrite Manning's law:

$$Q = \frac{1}{n}\frac{A^{5/3}(h)}{P^{2/3}(h)}\sqrt{S_0} = \frac{\sqrt{S_0}}{n}\frac{(A(h)/h)^{5/3}}{P^{2/3}(h)} \times h^{5/3},$$

where most of the variation with h is contained in the last term $h^{5/3}$, and by solving for that term we can re-write the equation in a form suitable for direct iteration

$$h = \left(\frac{Qn}{\sqrt{S_0}}\right)^{3/5} \times \frac{P^{2/5}(h)}{A(h)/h},$$

where the first term on the right is a constant for any particular problem, and the second term is expected to be a relatively slowly-varying function of depth, so that the whole right side varies slowly with depth – a primary requirement that the direct iteration scheme be convergent and indeed be quickly convergent.

Experience with typical trapezoidal sections shows that this works well and is quickly convergent. However, it also works well for flow in circular sections such as sewers, where over a wide range of depths the mean width does not vary much with depth either. For small flows and depths in sewers this is not so, and a more complicated method might have to be used.

Example: Calculate the normal depth in a trapezoidal channel of slope 0.001, Manning's coefficient $n = 0.04$*, width* $10\,\mathrm{m}$*, with batter slopes* $2:1$*, carrying a flow of* $20\,\mathrm{m}^3\,\mathrm{s}^{-1}$*. We have* $A = h\,(10 + 2\,h)$*,* $P = 10 + 4.472\,h$*, giving the scheme*

$$\begin{aligned} h &= \left(\frac{Qn}{\sqrt{S_0}}\right)^{3/5} \times \frac{(10 + 4.472\,h)^{2/5}}{10 + 2\,h} \\ &= 6.948 \times \frac{(10 + 4.472\,h)^{2/5}}{10 + 2\,h} \end{aligned}$$

and starting with $h = 2$ *we have the sequence of approximations: 2.000, 1.609, 1.639, 1.637 - quite satisfactory in its simplicity and speed.*

4.3 Conveyance

It is often convenient to use the conveyance K which contains all the roughness and cross-section properties, such that for steady uniform flow

$$Q = K\,\sqrt{S_0},$$

such that, using an electrical analogy, the flow (current) is given by a "conductance" (here conveyance) multiplied by a driving potential, which, here in this nonlinear case, is the square root of the bed slope. In more general non-uniform flows below we will see that we use the square root of the head gradient. With this definition, if we use Manning's law for the flow, K is defined by

$$K = \frac{1}{n} \times A\left(\frac{A}{P}\right)^{2/3} = \frac{1}{n} \times \frac{A^{5/3}}{P^{2/3}}, \tag{4.3}$$

where K is a function of the roughness and the local depth and cross-section properties. Textbooks often use conveyance to provide methods for computing the equivalent conveyance of compound sections such as that shown in Figure 4-2. However, for such cases where a river has overflowed its banks, the flow situation is much more likely to be more two-dimensional than one-dimensional. The extent of the various elemental areas and the Manning's roughnesses of the different parts are all such as to often render a detailed "rational" calculation unjustified.

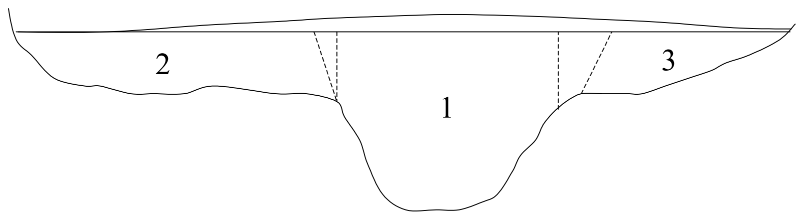

Figure 4-2.

In the compound channel in the figure, even though the surface might actually be curved as shown and the downstream slope and/or bed slope might be different across the channel, the tradition is that we assume it to be the same. The velocities in the individual sections are, in general, different. We write Manning's law for each section based on the mean bed slope:

$$Q_1 = K_1 S_0^{1/2},\ Q_2 = K_2 S_0^{1/2},\ Q_3 = K_3 S_0^{1/2}$$

In a general case with n sub-sections, the total discharge is

$$Q = \sum_{i=1}^{n} Q_i = \sum_{i=1}^{n} K_i S_0^{1/2} = S_0^{1/2} \sum_{i=1}^{n} K_i = S_0^{1/2} K$$

where we use the symbol K for the total conveyance:

$$K = \sum_{i=1}^{n} K_i = \sum_{i=1}^{n} \frac{A_i^{5/3}}{n_i P_i^{2/3}}.$$

5. Steady gradually-varied non-uniform flow

Steady gradually-varied flow is where the conditions (possibly the cross-section, but often just the surface elevation) vary slowly along the channel but do not change with time. The most common situation where this arises is in the vicinity of a *control* in a channel, where there may be a structure such as a weir, which has a particular discharge relationship between the water surface level and the discharge. Far away from the control, the flow may be uniform, and there the relationship between surface elevation and discharge is in general a different one, typically being given by Manning's law, (4.1). The transition between conditions at the control and where there is uniform flow is described by the *gradually-varied flow equation*, which is an ordinary differential equation for the water surface height. The solution will approach uniform flow if the channel is prismatic, but in general we can treat non-prismatic waterways also.

In sub-critical flow the flow is relatively slow, and the effects of any control can propagate back up the channel, and so it is that the numerical solution of the gradually-varied flow equation also proceeds in that direction. On the other hand, in super-critical flow, all disturbances are swept downstream, so that the effects of a control cannot be felt upstream, and numerical solution also proceeds downstream from the control.

Solution of the gradually-varied flow equation is a commonly-encountered problem in open channel hydraulics, as it is used to determine, for example, how far upstream water levels might be increased, and hence flooding enhanced, due to downstream works, such as the installation of a bridge.

5.1 Derivation of the gradually-varied flow equation

Consider the elemental section of waterway of length Δx shown in Figure 5-1. We have shown stations 1 and 2 in what might be considered the reverse order, but we will see that for the more common sub-critical flow, numerical solution of the governing equation will proceed back up the stream. Considering stations 1 and 2:

$$\begin{aligned} \text{Total head at 2} &= H_2 \\ \text{Total head at 1} &= H_1 = H_2 - H_L, \end{aligned}$$

and we introduce the concept of the *friction slope* S_f which is the gradient of the total energy line such that $H_L = S_f \times \Delta x$. This gives

$$H_1 = H_2 - S_f\,\Delta x,$$

and if we introduce the Taylor series expansion for H_1:

$$H_1 = H_2 + \Delta x\,\frac{dH}{dx} + \ldots,$$

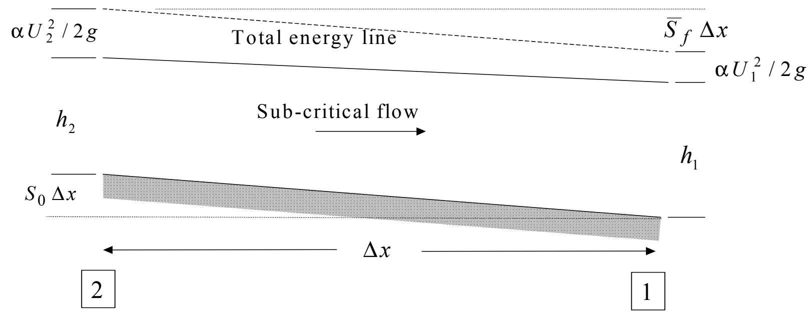

Figure 5-1. Elemental section of waterway

substituting and taking the limit $\Delta x \to 0$ gives

$$\frac{dH}{dx} = -S_f, \tag{5.1}$$

an ordinary differential equation for the head as a function of x.

To obtain the frictional slope, we use either of the frictional laws of Chézy or Manning (or a smooth-wall formula), where we make the assumption that the equation may be extended from uniform flow (where the friction slope equals the constant bed slope) to this non-uniform case, such that the discharge at any point is given by, for the case of Manning:

$$Q = \frac{1}{n}\frac{A^{5/3}}{P^{2/3}}\sqrt{S_f},$$

but where we have used the friction slope S_f rather than bed slope S_0, as in uniform flow. Solving for S_f: the friction slope is given by

$$S_f = \frac{Q^2}{K^2(h)}, \tag{5.2}$$

where we have used the conveyance K, which was defined in equation (4.3), but we repeat here,

$$K(h) = \frac{1}{n}\frac{A^{5/3}}{P^{2/3}},$$

showing the section properties to be a function of the local depth, where we have restricted our attention to prismatic channels on constant slope. This now means that for a given constant discharge we can write the differential equation (5.1) as

$$\frac{dH}{dx} = -S_f(h). \tag{5.3}$$

As we have had to use local depth on the right side, we have to show the head to be a function of depth h, so that we write

$$H = h + z_{\min} + \frac{\alpha}{2g}\frac{Q^2}{A^2(h)}. \tag{5.4}$$

Differentiating:

$$\frac{dH}{dx} = \frac{dh}{dx} + \frac{dz_{\min}}{dx} - \frac{\alpha}{g}\frac{Q^2}{A^3(h)}\frac{dA(h)}{dx}. \tag{5.5}$$

The derivative $dz_{\min}/dx = -S_0$, where S_0 is the bed slope, which we have defined to be positive for the usual case of a downwards-sloping channel. Now we have to express the $dA(h)/dx$ in terms of other quantities. In our earlier work we saw that if the surface changed by an amount Δh, then the change in area due to this was $\Delta A = B\,\Delta h$, and so we can write $dA(h)/dx = B\,dh/dx$, and substituting these results into equation (5.5) gives

$$\frac{dH}{dx} = -S_0 + \left(1 - \frac{\alpha}{g}\frac{Q^2 B(h)}{A^3(h)}\right)\frac{dh}{dx} = -S_0 + \left(1 - \alpha F^2(h)\right)\frac{dh}{dx},$$

where the Froude number has entered, shown here as a function of depth. Finally, substituting into (5.3) we obtain

$$\frac{dh}{dx} = \frac{S_0 - S_f(h)}{1 - \alpha F^2(h)} = \frac{S_0 - Q^2/K^2(h)}{1 - \alpha F^2(h)}, \tag{5.6}$$

a differential equation for depth h as a function of x, where on the right we have shown the functional dependence of the various terms. This, or the less-explicit form (5.3), are forms of the *gradually-varied flow equation*, from which a number of properties can be inferred.

5.2 Properties of gradually-varied flow and the governing equation

- The equation and its solutions are important, in that they tell us how far the effects of a structure or works in or on a stream extend upstream or downstream.
- It is an ordinary differential equation of first order, hence one boundary condition must be supplied to obtain the solution. In sub-critical flow, this is the depth at a downstream control; in super-critical flow it is the depth at an upstream control.
- In general that boundary depth is not equal to the normal depth, and the differential equation describes the transition from the boundary depth to normal depth – upstream for sub-critical flow, downstream for supercritical flow. The solutions look like exponential decay curves, and below we will show that they are, to a first approximation.
- If that approximation is made, the resulting analytical solution is useful in providing us with some insight into the quantities which govern the extent of the upstream or downstream influence.
- The differential equation is nonlinear, and the dependence on h is complicated, such that analytical solution is not possible without an approximation, and we will usually use numerical methods.
- The uniform flow limit satisfies the differential equation, for when $S_f = S_0$, $dh/dx = 0$, and the depth does not change.
- As the flow approaches critical flow, when $\alpha F^2 \to 1$, then $dh/dx \to \infty$, and the surface becomes vertical. This violates the assumption we made that the flow is gradually varied and the pressure distribution is hydrostatic. This is the one great failure of our open channel hydraulics at this level, that it cannot describe the transition between sub- and super-critical flow.

5.3 Classification system for gradually-varied flows

The differential equation can be used as the basis for a dual classification system of gradually-varied flows:

- one based on 5 conditions for slope, essentially as to how the normal depth compares with critical depth, and 3 conditions for the actual depth, and how it compares with both normal, and critical depths, as shown in the Table:

Slope classification	
Steep slope:	$h_n < h_c$
Critical slope:	$h_n = h_c$
Mild slope:	$h_n > h_c$
Horizontal slope:	$h_n = \infty$
Adverse slope:	h_n does not exist
Depth classification	
Zone **1**:	$h > h_n$ and h_c
Zone **2**:	h between h_n and h_c
Zone **3**:	$h < h_n$ and h_c

Figure 5-2 shows the behaviour of the various solutions. In practice, the most commonly encountered are the M_1, the backwater curve on a mild slope; M_2, the drop-down curve on a mild slope, and S_2, the drop-down curve on a steep slope.

5.4 Some practical considerations

5.4.1 Flood inundation studies

Figure 5-3 shows a typical subdivision of a river and its flood plain for a flood inundation study, where solution of the gradually-varied flow equation would be required. It might be wondered how the present methods can be used for problems which are unsteady, such as the passage of a substantial flood, where on the front face of the flood wave the water surface is steeper and on the back face it is less steep. In many situations, however, the variation of the water slope about the steady slope is relatively small, and the wavelength of the flood is long, so that the steady model can be used as a convenient approximation. The inaccuracies of knowledge of the geometry and roughness are probably such as to mask the numerical inaccuracies of the solution. Below we will present some possible methods and compare their accuracy.

5.4.2 Incorporation of losses

It is possible to incorporate the losses due, say, to a sudden expansion or contraction of the channel, such as shown in Figure 5-4. After an expansion the excess velocity head is destroyed through turbulence. Before an expansion the losses will not be so large, but there will be some extra losses due to the convergence and enhanced friction. We assume that the expansion/contraction head loss can be written

$$\Delta H_e = C\left(\frac{Q^2}{2gA_2^2} - \frac{Q^2}{2gA_1^2}\right),$$

where $C \approx 0.3$ for expansions and 0.1 for a contraction.

5.5 Numerical solution of the gradually-varied flow equation

Consider the gradually-varied flow equation (5.6)

$$\frac{dh}{dx} = \frac{S_0 - S_f(h)}{1 - \alpha F^2(h)},$$

where both $S_f(h) = Q^2/K^2(h)$ and $F^2(h) = Q^2 B(h)/gA^3(h)$ are functions of Q as well as the depth h. However as Q is constant for a particular problem we do not show the functional dependence on it. The equation is a differential equation of first order, and to obtain solutions it is necessary to have a boundary condition $h = h_0$ at a certain $x = x_0$, which will be provided by a control. The problem may be solved using any of a number of methods available for solving ordinary differential equations which

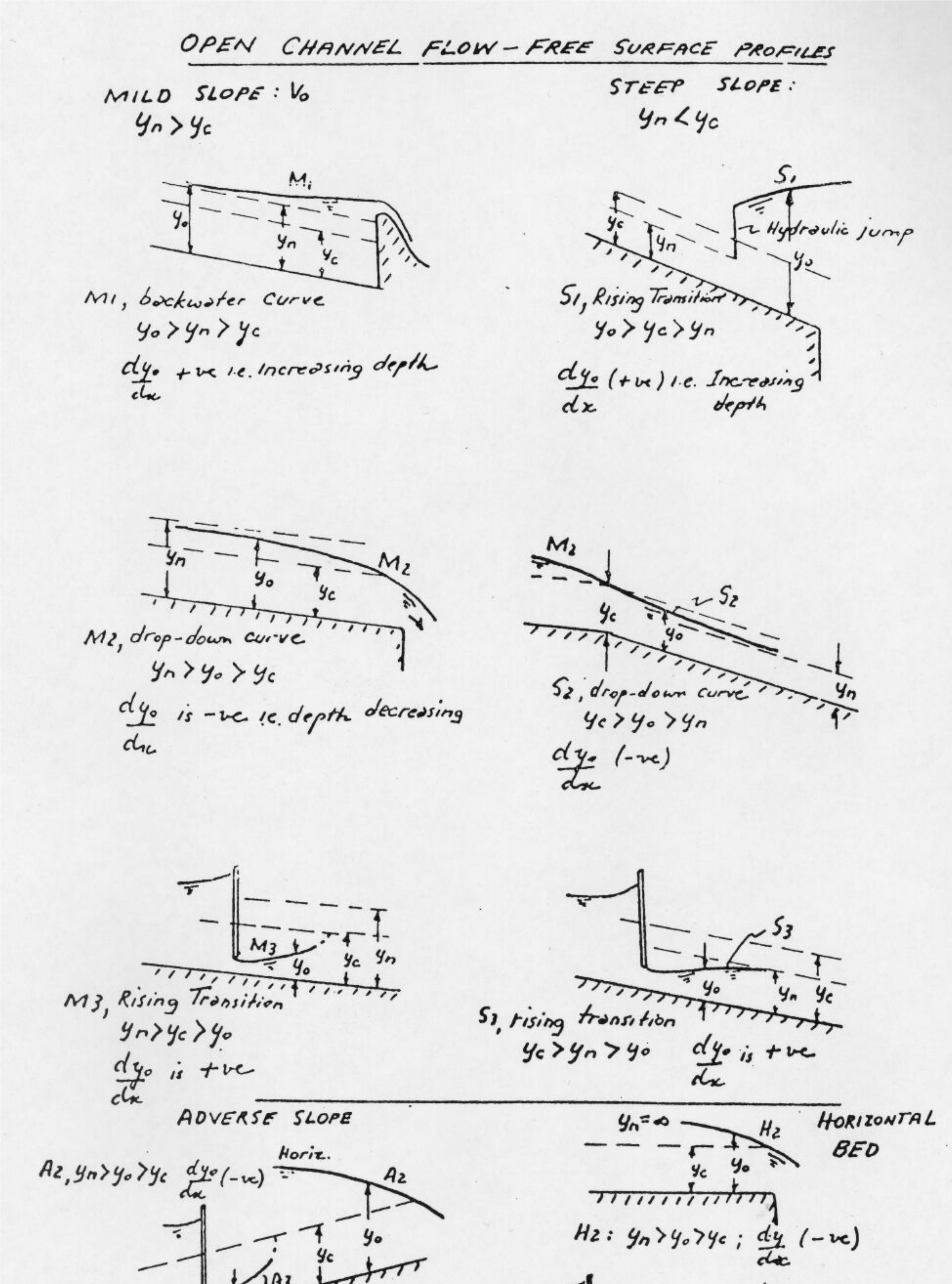

Figure 5-2. Typical gradually-varied flow surface profiles, drawn by Dr I. C. O'Neill.

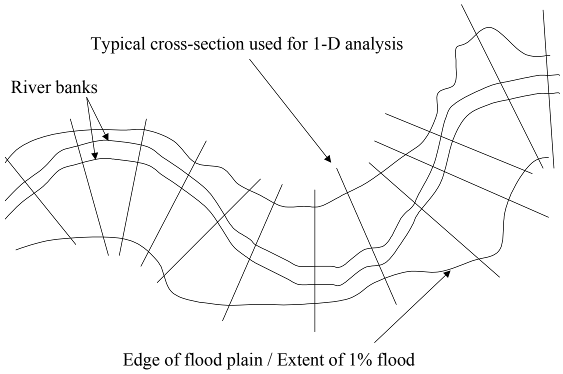

Figure 5-3. Practical river problem with subdivision

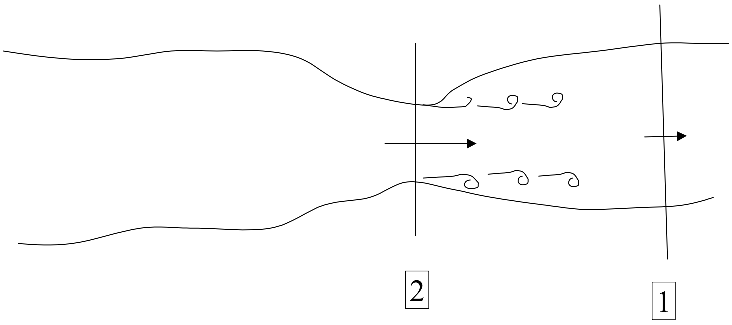

Figure 5-4. Flow separation and head loss due to a contraction

are described in books on numerical methods. These methods are usually accurate and can be found in many standard software packages. It is surprising that books on open channels do not recognise that the problem of numerical solution of the gradually-varied flow equation is actually a standard numerical problem, although practical details may make it more complicated. Instead, such texts use methods such as the "Direct step method" and the "Standard step method". There are several software packages such as HEC-RAS which use such methods, but solution of the gradually-varied flow equation is not a difficult problem to solve for specific problems in practice if one knows that it is merely the solution of a differential equation, and here we briefly set out the nature of such schemes.

The direction of solution is very important. If the different conventional cases in Figure 5-2 are examined, it can be seen that for the mild slope (sub-critical flow) cases that the surface decays somewhat exponentially to normal depth upstream from a downstream control, whereas for steep slope (super-critical flow) cases the surface decays exponentially to normal depth downstream from an upstream control. This means that to obtain numerical solutions we will always solve (a) for sub-critical flow: from the control *up*stream, and (b) for super-critical flow: from the control *down*stream.

5.5.1 Euler's method

The simplest (Euler) scheme to advance the solution from (x_i, h_i) to $(x_i + \Delta x_i, h_{i+1})$ is

$$x_{i+1} \approx x_i + \Delta x_i, \quad \text{where } \Delta x_i \text{ is negative for subcritical flow,} \tag{5.7}$$

$$h_{i+1} \approx h_i + \Delta x_i \left.\frac{dh}{dx}\right|_i = h_i + \Delta x_i \frac{S_0 - S_f(h_i)}{1 - \alpha F^2(h_i)}. \tag{5.8}$$

This is the simplest but least accurate of all methods – yet it might be appropriate for open channel problems where quantities may only be known approximately. One can use simple modifications such as Heun's method to gain better accuracy – or even more simply, just take smaller steps Δx_i.

5.5.2 Heun's method

In this case the value of h_{i+1} calculated from equation (5.8) is used as a first estimate h^*_{i+1}, then the value of the right hand side of the differential equation is also calculated there, and the mean of the two values taken. That is,

$$x_{i+1} \approx x_i + \Delta x_i, \quad \text{again where } \Delta x_i \text{ is negative for subcritical flow,} \tag{5.9}$$

$$h^*_{i+1} = h_i + \Delta x_i \frac{S_0 - S_f(h_i)}{1 - \alpha F^2(h_i)}, \tag{5.10}$$

$$h_{i+1} = h_i + \frac{\Delta x_i}{2}\left(\frac{S_0 - S_f(h_i)}{1 - \alpha F^2(h_i)} + \frac{S_0 - S_f(h^*_{i+1})}{1 - \alpha F^2(h^*_{i+1})}\right). \tag{5.11}$$

Neither of these two methods are presented in hydraulics textbooks as alternatives. Although they are simple and flexible, they are not as accurate as other less-convenient methods described further below. The step Δx_i can be varied at will, to suit possible irregularly spaced cross-sectional data.

5.5.3 Predictor-corrector method – Trapezoidal method

This is simply an iteration of the last method, whereby the step in equation (5.11) is repeated several times, at each stage setting h^*_{i+1} equal to the updated value of h_{i+1}. This gives an accurate and convenient method, and it is surprising that it has not been used.

5.5.4 Direct step method

Textbooks do present the Direct Step method, which is applied by taking steps in the height and calculating the corresponding step in x. It is only applicable to problems where the channel is prismatic. The reciprocal of equation (5.1) is

$$\frac{dx}{dH} = -\frac{1}{S_f},$$

which is then approximated by a version of Heun's method, but which is not a correct rational approximation:

$$\Delta x = -\frac{\Delta H}{\bar{S}_f}, \tag{5.12}$$

where a mean value of the friction slope is used. The procedure is: for the control point x_0 and h_0, calculate H_0 from equation (5.4), then assume a finite value of depth change Δh to compute $h_1 = h_0 + \Delta h$, from which H_1 is calculated from equation (5.4), giving $\Delta H = H_1 - H_0$. Then with $\bar{S}_f = (S_f(h_0) + S_f(h_1))/2$, equation (5.12) is used to calculate the corresponding Δx, giving $x_1 = x_0 + \Delta x$. The process is then repeated to give x_2 and h_2 and so on. It is important to choose the correct sign of Δh such that computations proceed in the right direction such that, for example, Δx is negative for sub-critical flow, and computations proceed upstream.

The method has the theoretical disadvantage that it is an inconsistent approximation, in that it should actually be computing the mean of $1/S_f$, namely $\overline{1/S_f}$, rather than $1/\bar{S}_f$. More importantly it has practical disadvantages, such that it is applicable only to prismatic sections, results are not obtained at specified points in x, and as uniform flow is approached the Δx become infinitely large. However it is a surprisingly accurate method.

5.5.5 Standard step method

This is an *implicit* method, requiring numerical solution of a transcendental equation at each step. It can be used for irregular channels, and is rather more general. In this case, the distance interval Δx is specified and the corresponding depth change calculated. In the Standard step method the procedure is to write

$$\Delta H = -S_f \, \Delta x,$$

and then write it as

$$H_2(h_2) - H_1(h_1) = -\frac{\Delta x}{2} \left(S_{f1} + S_{f2} \right),$$

for sections 1 and 2, where the mean value of the friction slope is used. This gives

$$\alpha \frac{Q^2}{2gA_2^2} + Z_2 + h_2 = \alpha \frac{Q^2}{2gA_1^2} + Z_1 + h_1 - \frac{\Delta x}{2} \left(S_{f1} + S_{f2} \right),$$

where Z_1 and Z_2 are the elevations of the bed. This is a transcendental equation for h_2, as this determines A_2, P_2, and S_{f2}. Solution could be by any of the methods we have had for solving transcendental equations, such as direct iteration, bisection, or Newton's method.

Although the Standard step method is an accurate and stable approximation, the lecturer considers it unnecessarily complicated, as it requires solution of a transcendental equation at each step. It would be much simpler to use a simple explicit Euler or Heun's method as described above.

Example: Consider a simple backwater problem to test the accuracy of the various methods. A trapezoidal channel with bottom width $W = 10\,\mathrm{m}$*, side batter slopes of 2:1, is laid on a slope of* $S_0 = 10^{-4}$*, and carries a flow of* $Q = 15\,\mathrm{m^3\,s^{-1}}$*. Manning's coefficient is* $n = 0.025$*. At the downstream control the depth is 2.5*m*. Calculate the surface profile (and how far the effect of the control extends upstream). Use 10 computational steps over a length of* $30\,\mathrm{km}$*.*

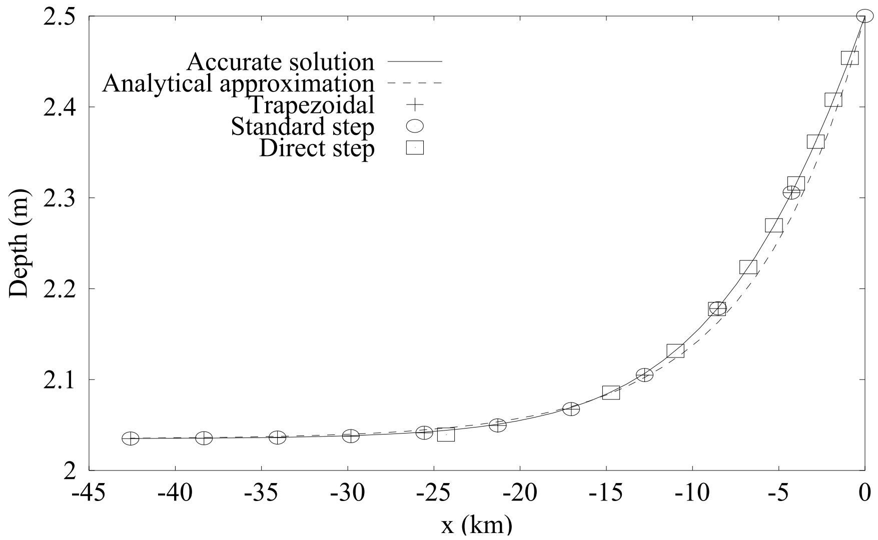

Figure 5-5. Comparison of different solution methods – depth plotted.

Figure 5-5 shows the results of the computations, where depth is plotted, while Figure 5-6 shows the same results, but where surface elevation is plotted, to show what the surface profile actually looks like. For relatively few computational points Euler's method was not accurate, and neither was Heun's method, and have not been plotted. The basis of accuracy is shown by the solid line, from a highly-accurate Runge-Kutta 4th order method. This is not recommended as a method, however, as it makes use of information from three intermediate points at each step, information which in non-prismatic channels is not available. It can be seen that the relatively simple Trapezoidal method is sufficiently accurate, certainly of acceptable practical accuracy. The Direct Step method was slightly more accurate, but the results show one of its disadvantages, that the distance between computational points becomes large as uniform flow is approached, and the points are at awkward distances. The last plotted point is at about $-25\,\mathrm{km}$; using points closer to normal depth gave inaccurate results. The Standard Step method was very accurate, but is not plotted as it is complicated to apply. Of course, if more computational points

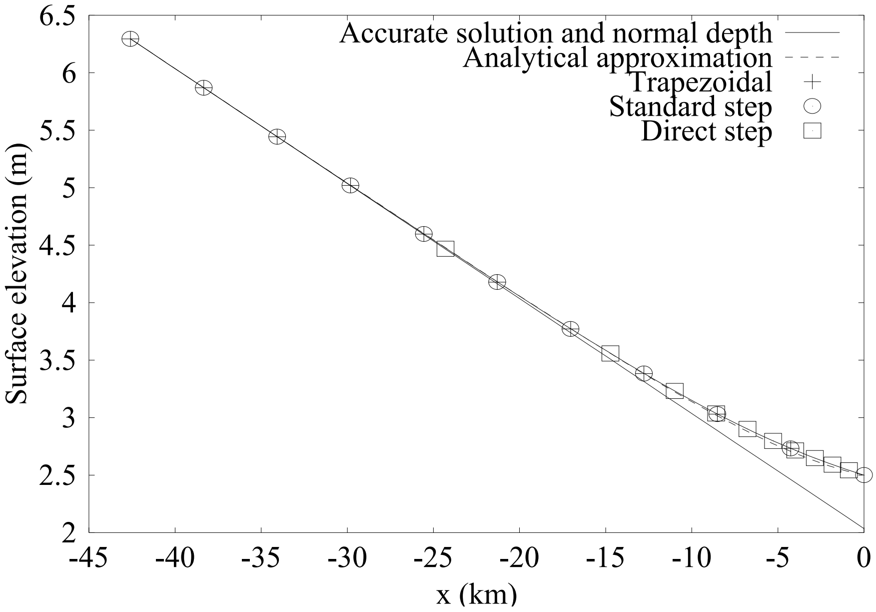

Figure 5-6. Comparison of different solution methods – elevation plotted.

were taken, more accurate results could be obtained. In this example we deliberately chose relatively few steps (10) so that the numerical accuracies of the methods could be compared.

Also plotted on the figures is a dotted line corresponding to the analytical solution which will be developed below. Although this was not as accurate as the numerical solutions, it does give a simple approximate result for the rate of decay and how far upstream the effects of the control extend. For many practical problems, this accuracy and simplicity may be enough.

The channel dimensions are typical of a large irrigation canal in the Murray Valley - it is interesting that the effects of the control extend for some 30km!

To conclude with a recommendation: the trapezoidal method, Heun's method iterated several times is simple, accurate, and convenient. If, however, a simple approximate solution is enough, then the following analytical solution can be used.

5.6 Analytical solution

Whereas the numerical solutions give us numbers to analyse, sometimes very few actual numbers are required, such as merely requiring how far upstream water levels are raised to a certain level, the effect of downstream works on flooding, for example. Here we introduce a different way of looking at a physical problem in hydraulics, where we obtain an approximate mathematical solution so that we can provide equations which reveal to us more of the nature of the problem than do numbers. Sometimes an *understanding* of what is important is more useful than numbers.

Consider the water surface depth to be written

$$h(x) = h_0 + h_1(x),$$

where we use the symbol h_0 for the constant normal depth, and $h_1(x)$ is a relatively small departure of the surface from the uniform normal depth. We use the governing differential equation (5.6) but we assume that the Froude number squared is sufficiently small that it can be ignored. This is not essential, but it makes the equations simpler to write and read. (As an example, consider a typical stream flowing at 0.5 m/s with a depth of 2m, giving $F^2 = 0.0125$ - there are many cases where F^2 can be neglected).

The simplified differential equation can be written

$$\frac{dh}{dx} = S_0 - S(h),$$

where for purposes of simplicity we have dropped the subscript f on the friction slope, now represented by S. Substituting our expansion, we obtain

$$\frac{dh_1}{dx} = S_0 - S(h_0 + h_1(x)). \tag{5.13}$$

Now we introduce the approximation that the h_1 term is relatively small such that we can write for the friction term its Taylor expansion about normal flow:

$$S(h_0 + h_1(x)) = S(h_0) + h_1(x) \times \frac{dS}{dh}(h_0) + \text{Terms proportional to } h_1^2.$$

We ignore the quadratic terms, write $dS/dh(h_0)$ as $S_0^{/}$, and substituting into equation (5.13), we obtain

$$\frac{dh_1}{dx} = -S_0^{/} h_1$$

where we have used $S(h_0) = S_0$. This is an ordinary differential equation which we can solve analytically. We have achieved this by "linearising" about the uniform flow. Now, by separation of variables we can obtain the solution

$$h_1 = Ge^{-S_0^{/} x},$$

and the full solution is

$$h = h_0 + Ge^{-S_0^{/} x}, \tag{5.14}$$

where G is a constant which would be evaluated by satisfying the boundary condition at the control. This shows that the water surface is actually approximated by an exponential curve passing from the value of depth at the control to normal depth. In fact, we will see that as $S_0^{/}$ is negative, far upstream as $x \to -\infty$, the water surface approaches normal depth.

Now we obtain an expression for $S_0^{/}$ in terms of the channel dimensions. From Manning's law,

$$S = n^2 Q^2 \frac{P^{4/3}}{A^{10/3}},$$

and differentiating gives

$$S^{/} = n^2 Q^2 \left(\frac{4}{3} \frac{P^{1/3}}{A^{10/3}} \frac{dP}{dh} - \frac{10}{3} \frac{P^{4/3}}{A^{13/3}} \frac{dA}{dh} \right),$$

which we can factorise, substitute $dA/dh = B$, and recognising the term outside the brackets, we obtain an analytical expression for the coefficient of x in the exponential function:

$$-S_0^{/} = n^2 Q^2 \frac{P_0^{4/3}}{A_0^{10/3}} \left(\frac{10}{3} \frac{B}{A} - \frac{4}{3} \frac{dP/dh}{P} \right)\bigg|_0 = S_0 \left(\frac{10}{3} \frac{B_0}{A_0} - \frac{4}{3} \frac{dP_0/dh_0}{P_0} \right).$$

The larger this number, the more rapid is the decay with x. The formula shows that more rapid decay occurs with steeper slopes (large S_0), smaller depths ($B_0/A_0 = 1/D_0$, where D_0 is the mean depth - if it decreases the overall coefficient increases), and smaller widths (P_0 is closely related to width, the term involving it can be written $d(\log P_0)/dh_0$: if P_0 decreases the term decreases - relatively slowly - but the negative sign means that the effect is to increase the magnitude of the overall coefficient). Hence, generally the water surface approaches normal depth more quickly for steeper, shallower and narrower (*i.e.* steeper and smaller) streams. The free surface will decay to 10% of its original departure from normal in a distance $X_{0.1}$, where

$$e^{-S_0^/ X_{0.1}} = 0.1.$$

In the example above, we obtain $-S_0^/ = 0.00015$,giving $X_{0.1} = -15400\,\mathrm{m}$, which seems roughly right when compared with the figure above. In a further interval of the same distance the surface will decay to 10% of this, 1% of its original magnitude, and so on. This theory has shown us something of the power of obtaining approximate solutions where no other analytical solutions exist, to provide us with a simpler understanding of the nature of the problem.

6. Unsteady flow

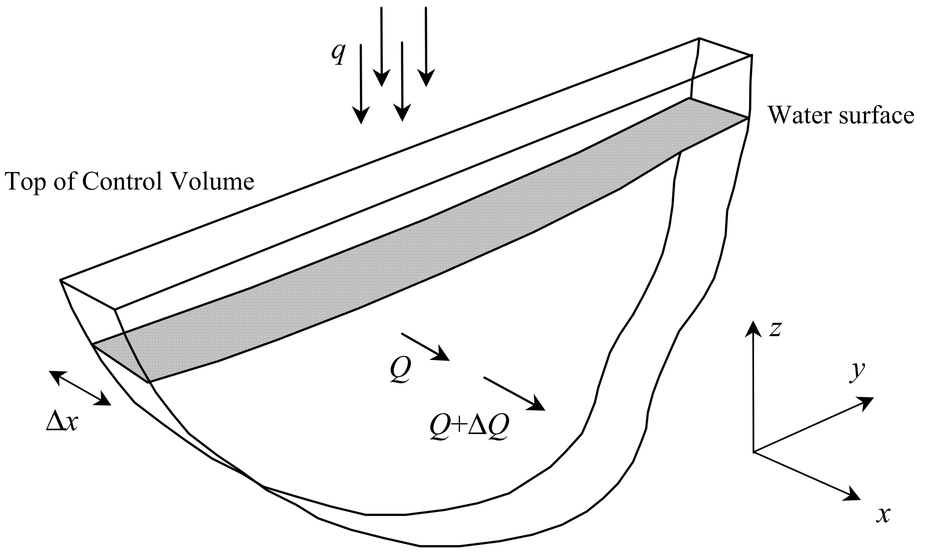

Figure 6-1. Element of non-prismatic waterway showing control volume extended into the air

We now consider the full equations for unsteady non-uniform flow. The fundamental assumption we make is that the flow is slowly varying along the channel. The mathematics uses a number of concepts from vector calculus, however we find that we can obtain general equations very powerfully, and the assumptions and approximations (actually very few!) are clear.

6.1 Mass conservation equation

Consider the elemental section of thickness Δx of non-uniform waterway shown in Figure 6-1, bounded by two vertical planes parallel to the $y - z$ plane. Consider also the control volume made up of this elemental section, but continued into the air such that the bottom and lateral boundaries are the river banks, and the upper boundary is arbitrary but never intersected by the water.

The Mass Conservation equation in integral form is, written for a control volume CV bounded by a control surface CS,

$$\underbrace{\frac{\partial}{\partial t}\int_{CV} \rho\, dV}_{\text{Total mass in CV}} + \underbrace{\int_{CS} \rho\, \mathbf{u}.\hat{\mathbf{n}}\, dS}_{\text{Rate of flow of mass across boundary}} = 0,$$

where t is time, dV is an element of volume, $\mathbf{u}$ is the velocity vector, $\hat{\mathbf{n}}$ is a unit vector with direction normal to and directed outwards from the control surface such that $\mathbf{u}.\hat{\mathbf{n}}$ is the component of velocity normal to the surface at any point, and dS is the elemental area of the control surface.

As the density of the air is negligible compared with the water, the domain of integration in the first integral reduces to the volume of water in the control volume, and considering the elemental slice,

$dV = \Delta x\, dA$, where dA is an element of cross-sectional area, the term becomes

$$\rho \Delta x \frac{\partial}{\partial t} \int_A dA = \rho \Delta x \frac{\partial A}{\partial t}$$

Now considering the second integral, on the upstream face of the control surface, $\mathbf{u}.\hat{\mathbf{n}} = -u$, where u is the x component of velocity, so that the contribution due to flow entering the control volume is

$$-\rho \int_A u\, dA = -\rho Q.$$

Similarly the downstream face contribution is

$$+\rho\,(Q + \Delta Q) = +\rho \left(Q + \frac{\partial Q}{\partial x} \Delta x \right).$$

On the boundaries which are the banks of the stream, the velocity component normal to the boundary is very small and poorly-known. We will include it in a suitably approximate manner. We lump this contribution from groundwater, inflow from rainfall, and tributaries entering the waterway, as a volume rate of q per unit length entering the stream. The rate at which mass enters the control volume is $\rho q \Delta x$ (*i.e.* an outflow of $-\rho q \Delta x$). Combining the contributions from the rate of change of mass in the CV and the net contribution across the two faces, and dividing by $\rho \Delta x$ we have the unsteady mass conservation equation

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = q. \tag{6.1}$$

Remarkably for hydraulics, this is an almost-exact equation - the only significant approximation we have made is that the waterway is straight! If we want to use surface elevation as a variable in terms of surface area, it is easily shown that in an increment of time δt if the surface changes by an amount $\delta \eta$, then the area changes by an amount $\delta A = B \times \delta \eta$, from which we obtain $\partial A / \partial t = B \times \partial \eta / \partial t$, and the mass conservation equation can be written

$$B \frac{\partial \eta}{\partial t} + \frac{\partial Q}{\partial x} = q. \tag{6.2}$$

The assumption that the waterway is straight has almost universally been made. Fenton & Nalder (1995)[1] have considered waterways curved in plan (*i.e.* most rivers!) and obtained the result (*cf.* equation 6.1):

$$\left(1 - \frac{n_m}{r} \right) \frac{\partial A}{\partial t} + \frac{\partial Q}{\partial s} = q,$$

where n_m is the transverse offset of the centre of the river surface from the curved streamwise reference axis s, and r is the radius of curvature of that axis. Usually n_m is small compared with r, and the curvature term is a relatively small one. It can be seen that if it is possible to choose the reference axis to coincide with the centre of the river viewed in plan, then $n_m = 0$ and curvature has no effect on this equation. This choice of axis is not always possible, however, as the geometry of the river changes with surface height.

6.2 Momentum conservation equation – the low inertia approximation

We can repeat the procedure above, but this time considering *momentum* conservation. It can be shown that we obtain

$$\frac{\partial Q}{\partial t} + \beta \frac{\partial}{\partial x} \left(\frac{Q^2}{A} \right) + gA \frac{\partial \eta}{\partial x} = -gAS_f. \tag{6.3}$$

Equations (6.2) and (6.3) are the *long wave* or *Saint-Venant* equations. They are used to simulate wave

[1] Fenton, J. D. & Nalder, G. V. (1995), Long wave equations for waterways curved in plan, *in Proc. 26th Congress IAHR, London*, Vol. 1, pp. 573–578.

motions in rivers and canals, notably the propagation of flood waves and the routine simulation of irrigation channel operations. There is a software industry which specialises in numerical solutions. It can be shown mathematically that solutions of these equations look like two families of waves, propagating at two different velocities, one upstream, the other downstream, with differing amounts of diffusive dissipation. The upstream propagating waves show rather more dissipation. The propagation in both directions is important in situations where transients are rapid, such as in hydro-electric supply canals. In most situations, however, the flow velocity is relatively small, such that a large simplification is possible. We can show that the first two terms in equation (6.3), relative to the others, are of the order of magnitude of F^2, and can be ignored.

Firstly consider $\partial Q/\partial t$. The discharge Q is of an order of magnitude $V \times W \times D$, where V is a typical velocity, W is a typical width, and D is a typical mean depth. The time scale of motion is given by L/V, where L is a typical length scale of the wave motion down the waterway, and our velocity scale gives a measure of how quickly it is swept past. Hence,

$$\frac{\partial Q}{\partial t} \text{ is of a scale } \frac{V \times W \times D}{L/V} = \frac{V^2 W D}{L}.$$

Now we examine the scale of the term $gA\partial\eta/\partial x$, and here we assume that the vertical scale of our disturbances is the vertical scale of the channel D, giving

$$gA\frac{\partial \eta}{\partial x} \text{ is of a scale } g \times W \times D \times \frac{D}{L} = \frac{gWD^2}{L}.$$

Now we compare the magnitudes of the two terms $\partial Q/\partial t : gA\partial\eta/\partial x$ and we find that the ratio of the two terms is of a scale

$$\frac{V^2 W D}{L} \times \frac{L}{gWD^2} = \frac{V^2}{gD}, \quad \text{the scale of the Froude number squared.}$$

Hence, possibly to our surprise, we find that the relative magnitude of the term $\partial Q/\partial t$ is roughly F^2, and in many flows in rivers and canals this is a small quantity and terms of this size can be ignored. Examining the second term in equation (6.3) it might be more obvious that it too is also of order F^2. If we neglect both such terms of order F^2, making the "low-inertia" approximation, we find that the momentum equation (6.3) can simply be approximated by

$$\frac{\partial \eta}{\partial x} + S_f = 0, \tag{6.4}$$

which expresses the fact that, even in a generally unsteady situation, the surface slope and the friction slope are the same magnitude. Now we use an empirical friction law for the friction slope S_f in terms of conveyance K, so that we write

$$S_f = \frac{Q^2}{K^2},$$

where the dependence of K on depth at a section would be given by Manning's law. Substituting this into (6.4) gives us an accurate approximation for the discharge in terms of the slope:

$$Q = K\sqrt{-\frac{\partial \eta}{\partial x}}, \tag{6.5}$$

even in a generally unsteady flow situation, provided the Froude number is sufficiently small. (Note that $\partial\eta/\partial x$ is always negative in situations where this theory applies!). This provides us with a good method of measuring the discharge - if we can calibrate a gauging station to give the conveyance as a function of surface height, then by measuring the surface slope we can get the discharge.

At this point it is easier to introduce the local depth h such that if Z is the local elevation of the bottom,

$$\eta = Z + h \quad \text{and} \quad \frac{\partial \eta}{\partial x} = \frac{\partial Z}{\partial x} + \frac{\partial h}{\partial x} = \frac{\partial h}{\partial x} - S_0,$$

in which case equation (6.5) can be written

$$Q = K\sqrt{S_0 - \frac{\partial h}{\partial x}}, \tag{6.6}$$

6.3 Diffusion routing and nature of wave propagation in waterways

Now we eliminate the discharge Q from the equations by simply substituting equation (6.6) into the mass conservation equation (6.2), noting that as the bed does not move, $\partial\eta/\partial t = \partial h/\partial t$, to give the single partial differential equation in the single variable h: ,

$$\frac{\partial h}{\partial t} + \frac{1}{B}\frac{\partial}{\partial x}\left(K\sqrt{S_0 - \frac{\partial h}{\partial x}}\right) = 0 \tag{6.7}$$

The conveyance is usually expressed as a function of roughness and of the local depth h, so that we can perform the differentiation in equation (6.7) and we assume that if the variation of the local depth is small compared with the overall slope so that $|\partial h/\partial x| \ll S_0$ we can write

$$\frac{\partial h}{\partial t} + \underbrace{\frac{\sqrt{S_0}}{B}\frac{dK}{dh}}_{\text{Propagation velocity}}\frac{\partial h}{\partial x} = \underbrace{\frac{K}{2B\sqrt{S_0}}}_{\text{Diffusion coefficient}}\frac{\partial^2 h}{\partial x^2} \tag{6.8}$$

We now have a rather simpler single equation in a single unknown. This is an *advection-diffusion equation*, and the nature of it is rather clearer than our original pair of equations. It has solutions which propagate at the propagation velocity shown. We write the equation as

$$\frac{\partial h}{\partial t} + c\frac{\partial h}{\partial x} = \nu\frac{\partial^2 h}{\partial x^2}, \tag{6.9}$$

where c is a propagation speed, the *kinematic wave speed*, and ν is a diffusion coefficient (with units of $\mathrm{L}^2\mathrm{T}^{-1}$), given by

$$c = \frac{\sqrt{S_0}}{B}\frac{dK}{dh} \quad \text{and} \quad \nu = \frac{K}{2B\sqrt{S_0}}. \tag{6.10}$$

As $K = 1/n \times A^{5/3}(h)/P^{2/3}(h)$, we can differentiate to give the kinematic wave speed c for an arbitrary section. However for the purposes of this course we can consider a wide rectangular channel ($h \ll B$) such that $A \approx bh$ and $P \approx B = b$ such that $K = 1/n \times Bh^{5/3}$. Differentiating, $dK/dh = 1/n \times 5/3 \times Bh^{2/3}$, and substituting into (6.10) we obtain

$$c = \frac{5}{3} \times \frac{\sqrt{S_0}}{n}h^{2/3}.$$

Now, the velocity of flow in this waterway is $U = 1/n \times (A/P)^{2/3}\sqrt{S_0}$, which for our wide-channel approximation is $U \approx \sqrt{S_0}/n \times h^{2/3}$, and so we obtain the approximate relationship for the speed of propagation of disturbances in a wide channel:

$$c \approx \frac{5}{3}U, \tag{6.11}$$

which is a very simple expression: the speed of propagation of disturbances is approximately $1\frac{2}{3}$ times the mean speed of the water.

In all text books another wave velocity is often presented, which is $c = \sqrt{gA/B}$, where A/B is the mean depth of the water. This is obtained from dynamical considerations of long waves in still water. It is indeed the order of magnitude of the speed at which waves do move over essentially still water, and is widely used, incorrectly, to estimate the speed of disturbances in rivers and canals. It is called the *dynamic wave speed*, and part of waves do travel at this speed. However, provided the Froude number is small, such that $F^2 \ll 1$, equation (6.11) is a good approximation to the speed at which the bulk of disturbances propagate.

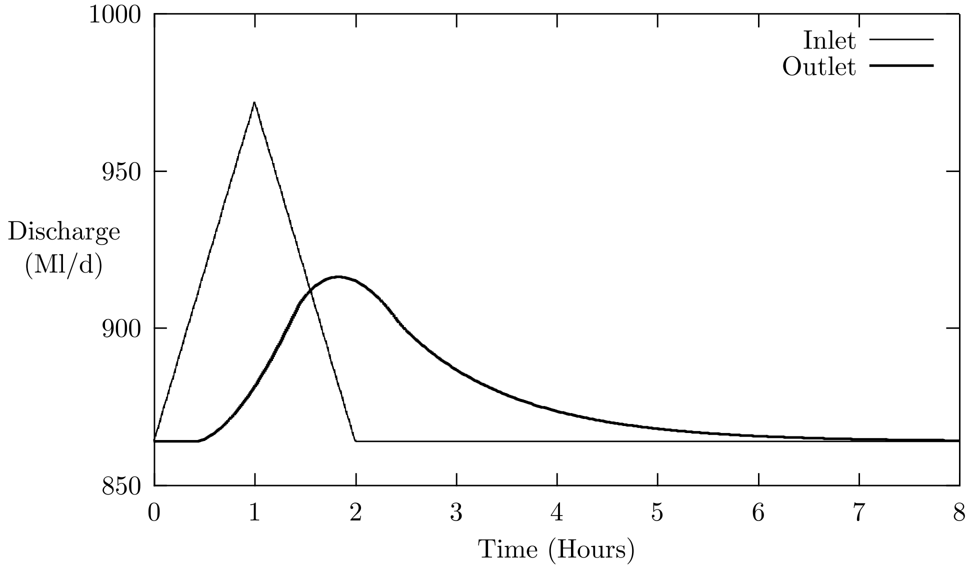

Figure 6-2. Inflow and outflow hydrographs for short pool on a shallow slope showing the effects of friction

Now to consider the effects of diffusion, if we examine the real or simulated propagation of waves in streams, the apparent motion is of waves propagating at the kinematic wave speed, but showing marked diminution in size as they propagate. We consider as a test case, a pool 7km long, bottom width of 7m, batter slopes of 1.5:1, a longitudinal slope of 0.0001, a depth of 2.1m at the downstream gate, and Manning's $n = 0.02$. We consider a base flow of 10 $\mathrm{m^3s^{-1}}$ increased linearly by 25% up to a maximum of 12.5 $\mathrm{m^3s^{-1}}$ and back down to the base flow over a period of two hours. A computer program simulated conditions in the canal. The results presented in Figure 6-2 show that over the length of only 7km the peak discharge has decreased by about 50% and has spread out considerably in time - the wave propagation is not just a simple translation. In fact, the peak takes about 55 minutes to traverse the pool, whereas using the dynamic wave speed that time would be 30 minutes, while simply using the kinematic wave speed it would be 140 minutes. The wave has diffused considerably, showing that simple deductions based on a wave speed are only part of the picture. This difference might be important for flood warning operations.

It is important to find out more about the real nature of wave propagation in waterways. Here we provide a simple tool for estimating the relative importance of diffusion. If we were to scale the advection-diffusion equation (6.9) such that it was in terms of a dimensionless variable x/L which would be of order of magnitude 1, then the ratio of the importance of the diffusion term to the advection term can be shown to be ν/cL. This looks like the inverse of a Reynolds number (which is correct – the Reynolds number is the inverse of a dimensionless viscosity or diffusion number). Now we substitute in the approximations for a wide channel, giving:

$$
\begin{aligned}
\text{A measure of the importance of diffusion} &= \frac{\nu}{cL} = \frac{K}{2B\sqrt{S_0}} \times \frac{B}{\sqrt{S_0}K'(h)\,L} \\
&= \frac{K/K'(h)}{2\,S_0 L}, \quad \text{and as } K \propto h^{5/3} \text{ this gives} \\
&\approx \frac{3}{10} \times \frac{h}{S_0 L}.
\end{aligned}
$$

This is a useful result, for it shows us the effects of diffusion very simply, as h is the depth of the stream, and $S_0 L$ is the amount by which a stream drops over the reach of interest, we have

$$
\text{A measure of the importance of diffusion} \approx \frac{3}{10} \times \frac{\text{Depth of stream}}{\text{Drop of stream}}.
$$

For the example above, this is about 0.8, showing that diffusion is as important as advection, and reminding us that the problem is not the simple translation of a wave.

This result is also interesting in considering different types of streams – a steep shallow mountain stream will show little diffusion, whereas a deep gently sloping stream will have marked diffusion!

7. Structures in open channels and flow measurement

The main texts to which reference can be made are Ackers, White, Perkins & Harrison (1978), French (1985), Henderson (1966), Novak (2001). Another useful scholarly reference is that by Jaeger (1956).

7.1 Overshot gate - the sharp-crested weir

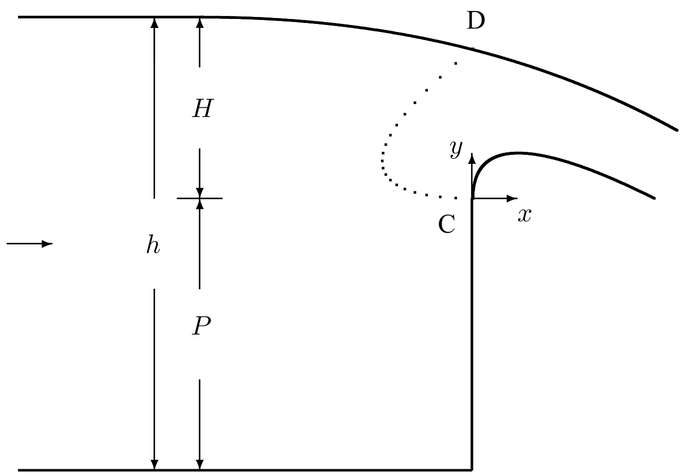

Figure 7-1. Side view of sharp-crested weir. The dotted line shows the notional pressure distribution above the crest.

The calculation of the discharge over a sharp-crested weir is one of the great misleading results in hydraulic engineering, which is widely known to be so, yet every textbook reproduces it. Instead of the actual pressure distribution shown dotted in Figure 7-1, the pressure is assumed to be zero over the crest, then Bernoulli's law applied, and then it is assumed that the fluid velocity is horizontal only (whereas it is actually vertical at the crest). It is necessary to introduce a coefficient of discharge which has a value of $0.6 - 0.7$ to make the theory work.

A more correct way to proceed is to use dimensional analysis – firstly to consider a weir of infinite length and in water of infinite depth. The head over the weir is H (as shown in Figure 7-1), gravitational acceleration is g, the discharge is q per unit length. We have three quantities, two dimensions (L and T), and so by the Buckingham π theorem, we have only a single quantity affecting results, which must therefore be a constant. Collecting the terms in the only possible dimensionless combination, we have

$$\frac{q}{\sqrt{gH^3}} = C = \text{constant},$$

hence we obtain the well-known $3/2$ law:

$$q = C\sqrt{g}H^{3/2}.$$

If we now consider a finite width of channel B, finite width of weir b, and a finite apron height P, with now a total discharge Q we obtain

$$\frac{Q}{b\sqrt{gH^3}} = f(b/B, H/P, b/P).$$

In fact, an approximate expression can be written

$$Q = C\sqrt{g}\, b\, H^{3/2}, \tag{7.1}$$

where C is a dimensionless coefficient. There is relatively little variation about a constant value of C, and a useful approximate expression is

$$Q = 0.6\sqrt{g}b\ (\eta_u - z_c)^{3/2}$$

where η_u is the upstream surface elevation and z_c is the elevation of the crest.

7.2 Triangular weir

An analysis as misguided as the traditional one produces the result quoted by French (1985) on p352, where some results of Bos (1978) are quoted. The expression is

$$Q = C\frac{8}{15}\sqrt{2g}\tan\frac{\theta}{2}H^{2.5},$$

where C is the coefficient of discharge, and where θ is the angle between the sides of the triangle. A typical result from French (1985) is that C is roughly 0.58, which has been found to agree well with experiment. We prefer to write the expression as

$$Q = 0.44\sqrt{g}\tan\frac{\theta}{2}\,H^{2.5}.$$

7.3 Broad-crested weirs – critical flow as a control

In this case theory is rather more applicable, and we have already considered that in this course. The flow upstream is subcritical, but the flow in the structure is made to reach critical - the flow reaches its critical depth at some point on top of the weir, and the weir provides a control for the flow. In this case, the head upstream (the height of the upstream water surface above the sill) uniquely determines the discharge, and it is enough to measure the upstream surface elevation where the flow is slow and the kinetic part of the head negligible to provide a point on a unique relationship between that head over the weir and the discharge. No other surface elevation need be measured. Such a horizontal flow control is called a *broad-crested weir*. We can apply simply theory of critical flows to give the discharge per unit

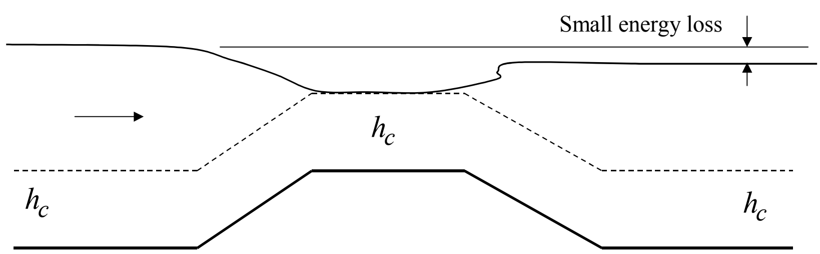

Figure 7-2. Broad-crested weir in a canal showing substantial recovery of energy

width q:

$$q = \sqrt{gh_c^3} = \sqrt{g\left(\tfrac{2}{3}H_c\right)^3} = \sqrt{\tfrac{8}{27}g}H_c^{3/2}$$

where

$$H_c = H_1 - \Delta = h_1 + \alpha\frac{U_1^2}{2g} - \Delta = h_1 + \frac{\alpha Q^2}{2gA_1^2} - \Delta.$$

As A_1, the upstream area, is a function of h_1, the relationship for discharge is not simply a $3/2$ power law, but for sufficiently low upstream Froude number the deviation will not be large. In practice the discharge for the weir as a whole is written, introducing various discharge coefficients Montes (1998, p250),

$$Q = 0.54 C_s C_e C_v b \sqrt{g} h_1^{3/2}.$$

7.4 Free overfall

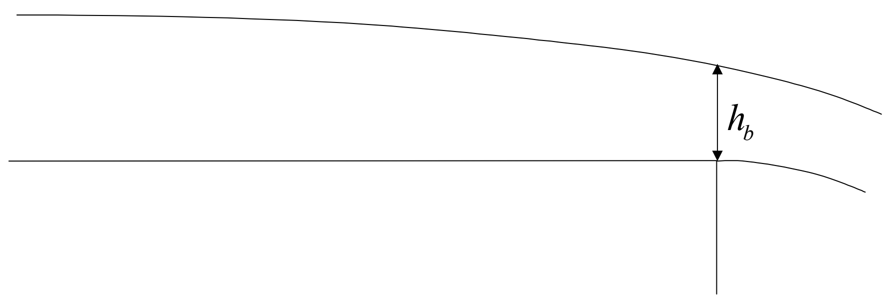

Figure 7-3. Free overfall

Consider the free overfall shown in Figure 7-3. It can be shown that the discharge per unit width is

$$q = 1.65\sqrt{g} h_b^{3/2}.$$

7.5 Undershot sluice gate

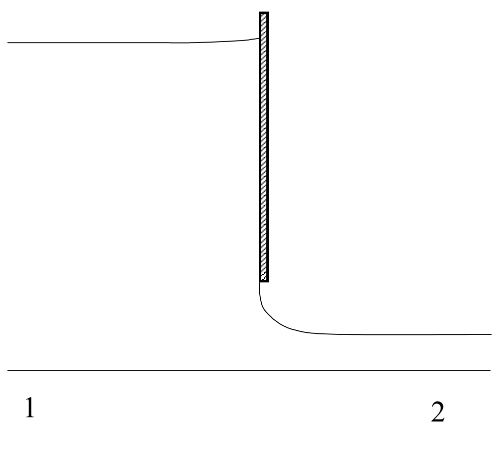

Figure 7-4. Sluice gate

Consider the case where the gate is not drowned, but a stream of supercritical flow can exist for some distance downstream. Applying the energy theorem between a point at section 1 and one at 2:

$$h_1 + \frac{\alpha_1}{2g}\left(\frac{q}{h_1}\right)^2 = h_2 + \frac{\alpha_2}{2g}\left(\frac{q}{h_2}\right)^2.$$

Solving for q (and assuming $\alpha_1 = \alpha_2 = \alpha$) we obtain:

$$q = \frac{1}{\sqrt{\alpha}} \frac{\sqrt{2g} h_1 h_2}{\sqrt{h_1 + h_2}},$$

which it is easier to write in dimensionless form:

$$\frac{q}{\sqrt{2gh_1^3}} = \frac{1}{\sqrt{\alpha}} \frac{h_2/h_1}{\sqrt{1 + h_2/h_1}}. \tag{7.2}$$

In practice we know the upstream depth h_1 and the gate opening h, such that

$$h_2 = C_c h,$$

where C_c is the coefficient of contraction. A number of theoretical and experimental studies have been made of this, but the variation is not large. Henderson (1966, p205) shows some of these but ends up recommending a constant value of 0.61.

Equation (7.2) is a convenient way of representation, as h_1 is given, and so we find that the dimensionless discharge is simply a function of h_2/h_1, the depth ratio, or h/h_1. We can approximate the result by noting that h_2/h_1 is small, and using $\alpha = 1$,

$$\frac{q}{\sqrt{2gh_1^3}} = \frac{h_2/h_1}{\sqrt{1 + h_2/h_1}} \approx \frac{h_2}{h_1},$$

and we find that to first order, for small gate openings,

$$q \approx \sqrt{2gh_1} \times h_2,$$

and the result is almost obvious, that the discharge per unit width is given by a velocity corresponding to the full upstream head of water multiplied by the downstream depth of flow.

7.6 Drowned undershot gate

In this case the analysis is rather more complicated, but we apply energy and momentum considerations where appropriate and find that we can extract a useful result.

7.7 Dethridge Meter

Developed in Australia, these have been almost universally used in our irrigation industry. They are the familiar water-wheels one sees in irrigation areas. They are now being replaced by more sophisticated methods.

8. The measurement of flow in rivers and canals

We have considered each of the major type of structures separately, as they are used for control of flows in channels, as well as measurement. Now we briefly describe a variety of methods which are used to measure flow without the use of structures, such that there is no obstacle to the flow. Then we consider how the movement of waves in rivers might affect the results, how the results are analysed and used.

8.1 Methods which do not use structures

1. **Velocity area method (”current meter method”):** The area of cross-section is determined from soundings, and flow velocities are measured using propeller current meters, electromagnetic sensors, or floats. The mean flow velocity is deduced from points distributed systematically over the river cross-section. In fact, what this usually means is that two or more velocity measurements are made on each of a number of vertical lines, and any one of several empirical expressions used to

calculate the mean velocity on each vertical, the lot then being integrated across the channel. One of the most common numerical methods used by hydrographers (the "Mean-Section Method") is wrong and should not be used. Rather, the Trapezoidal rule should be used, which is just as easily implemented. In a gauging in which the lecturer participated, a flow of 1693 Ml/d was calculated using the Mean-Section Method. Using the Trapezoidal rule, the flow calculated was 1721 Ml/d, a difference of 1.6%. Although the difference was not great, practitioners should be discouraged from using a formula which is wrong. The lecturer has developed a family of new methods for more accurate implementation of this method (see Fenton 2002).

2. **Stage-Discharge Method:** Here a *stage-discharge* relationship or *rating curve* is built up, typically using the velocity-area method, so that the functional relationship $Q_{\mathrm{r}}(\eta)$ can be determined, where Q_{r} is the rated discharge. Subsequent measurements of the surface elevation at some time t, such as a daily measurement, are then used to give the discharge:

$$Q(t) = Q_{\mathrm{r}}(\eta(t)).$$

This is very widely used and is the routine method of flow measurement. It will be described below.

3. **Slope-Stage-Discharge Method:** The method is presented in some books and in International and Australian Standards, however, especially in the latter, the presentation is confusing and at a low level, where no reference is made to the fact that underlying it the *slope* is being measured. Instead, the *fall* is described, which is the change in surface elevation between two surface elevation gauges and is simply the slope multiplied by the distance between them. No theoretical justification is provided and it is presented in a phenomenological sense (see, for example, Herschy 1995). An exception is Boiten (2000), however even that presentation loses sight of the pragmatic nature of determining a *stage-conveyance* relationship, and instead uses Manning's law in its classical form

$$Q = \frac{1}{n} \frac{A^{5/3}}{P^{2/3}} \sqrt{S_\eta},$$

where S_η is the free surface slope, $S_\eta = -\partial\eta/\partial x$, and it is assumed that the discharge must be given using these precise geometrical quantities of A and P. The lecturer suggests a rather more empirical approach, developing a *stage-conveyance* relationship by measuring the stage *and the slope* and using the relationship $Q = K(\eta)\sqrt{S_\eta}$,where $K(\eta)$ is the conveyance, to determine a particular value of K for measured values of stage and slope. Then, routinely, if the two were measured, discharge would be calculated in the form

$$Q(t) = K(\eta(t))\sqrt{S_\eta(t)}.$$

4. **Ultrasonic flow measurement:** This is a method used in the irrigation industry in Australia, but is

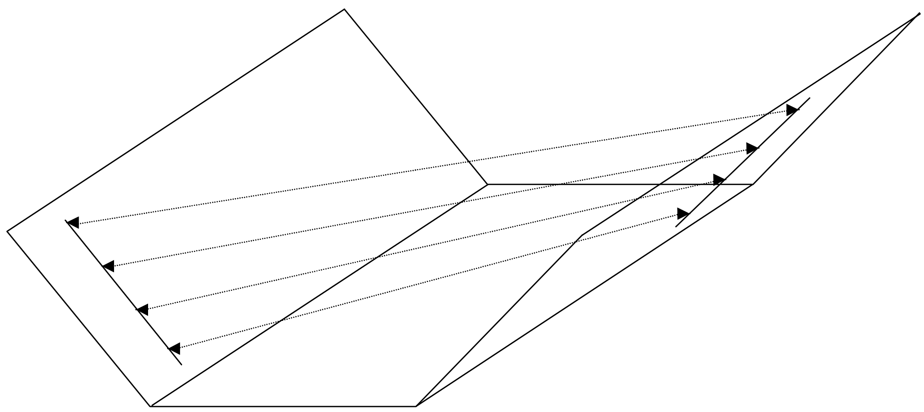

Figure 8-1. Array of four ultrasonic beams in a channel

also being used in rivers in the USA. Consider the situation shown in Figure 8-1, where some three or four beams of ultrasonic sound are propagated diagonally across a stream at different levels. The time of travel of sound in one direction is measured, as is the time in the other. The difference can be used to compute the mean velocity along that path, *i.e.* at that level. These values then have to be integrated in the vertical. Commercial implementations of the latter process are woefully inadequate, and installations of this type of meter do not gain the accuracy of which they are easily capable.

5. **Electromagnetic methods:** The motion of water flowing in an open channel cuts a vertical magnetic field which is generated using a large coil buried beneath the river bed, through which an electric current is driven. An electromotive force is induced in the water and measured by signal probes at each side of the channel. This very small voltage is directly proportional to the average velocity of flow in the cross-section. This is particularly suited to measurement of effluent, water in treatment works, and in power stations, where the channel is rectangular and made of concrete; as well as in situations where there is much weed growth, or high sediment concentrations, unstable bed conditions, backwater effects, or reverse flow. This has the advantage that it is an integrating method, however in the end recourse has to be made to empirical relationships between the measured electrical quantities and the flow.

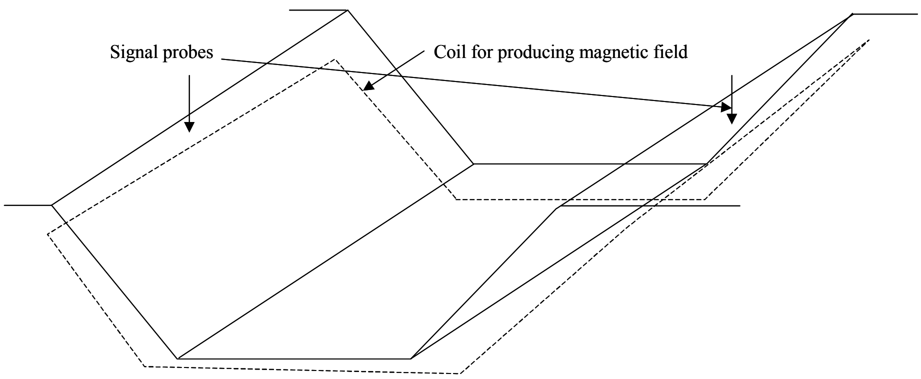

Figure 8-2. Electromagnetic installation, showing coil and signal probes

6. **Acoustic-Doppler Current Profiling methods:** In these, a beam of sound of a known frequency is transmitted into the fluid, often from a boat. When the sound strikes moving particles or regions of density difference moving at a certain speed, the sound is reflected back and received by a sensor mounted beside the transmitter. According to the Doppler effect, the difference in frequency between the transmitted and received waves is a direct measurement of velocity. In practice there are many particles in the fluid and the greater the area of flow moving at a particular velocity, the greater the number of reflections with that frequency shift. Potentially this method is very accurate, as it purports to be able to obtain the velocity over quite small regions and integrate them up. However, this method does not measure in the top 15% of the depth or near the boundaries, and the assumption that it is possible to extract detailed velocity profile data from a signal seems to be optimistic. The lecturer remains unconvinced that this method is as accurate as is claimed.

7. **Slope area method:** This is used to calculate peak discharges after the passage of a flood. An ideal site is a reach of uniform channel in which the flood peak profile is defined on both banks by high water marks. From this information the slope, the cross-sectional area and wetted perimeter can be obtained, and the discharge computed with the Manning formula or the Chézy formula. To do this however, roughness coefficients must be known, such as Manning's n in the formula

$$Q = \frac{1}{n}\frac{A^{5/3}}{P^{2/3}}\sqrt{\bar{S}},$$

where A is the area, P the perimeter, and $\bar{S}$ the slope. It is approximate at best.

8. **Dilution methods** In channels where cross-sectional areas are difficult to determine (*e.g.* steep

mountain streams) or where flow velocities are too high to be measured by current meters *dilution* or *tracer* methods can be used, where continuity of the tracer material is used with steady flow. The rate of input of tracer is measured, and downstream, after total mixing, the concentration is measured. The discharge in the stream immediately follows.

9. **Integrating float methods** There is another rather charming and wonderful method which has been very little exploited. At the moment it has the status of a single measurement method, however the lecturer can foresee it being developed as a continuing method. Consider a single buoyant particle (a float, an orange, an air bubble), which is released from a point on the bed. We assume that it has a constant rise velocity w. As it rises it passes through a variable horizontal velocity field $u(z)$, where z is the vertical co-ordinate, it samples the horizontal velocity equally at each point. The point where the particle reaches the surface downstream of the point at which it was released on the bed can be shown to be directly proportional to the mean horizontal velocity experienced on its vertical traverse. Now, if we were to release bubbles from a pipe across the bed of the stream, on the bed, then the expression

$$Q = \text{Bubble rise velocity } \times \text{area on surface between bubble path pattern and line of release}$$

is possibly the most direct and potentially the most accurate of all flow measurement methods!

8.2 The hydraulics of a gauging station

Almost universally the routine measurement of the state of a river is that of the *stage*, the surface elevation at a gauging station, usually specified relative to an arbitrary local datum. While surface elevation is an important quantity in determining the danger of flooding, another important quantity is the actual flow rate past the gauging station. Accurate knowledge of this instantaneous discharge - and its time integral, the total volume of flow - is crucial to many hydrologic investigations and to practical operations of a river and its chief environmental and commercial resource, its water. Examples include decisions on the allocation of water resources, the design of reservoirs and their associated spillways, the calibration of models, and the interaction with other computational components of a network. A typical set-up

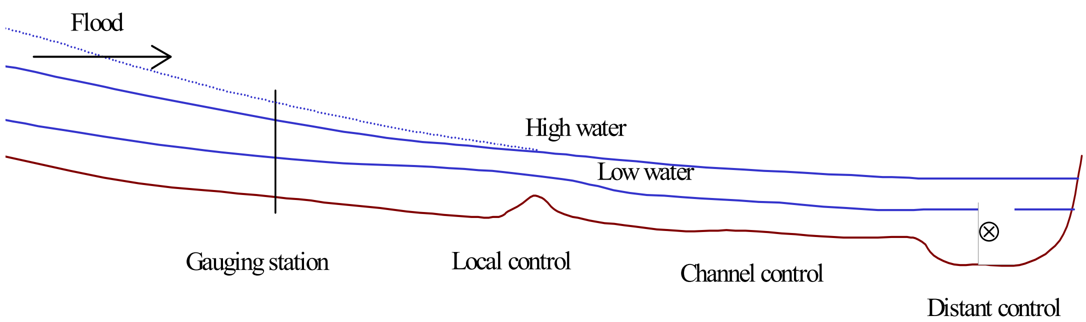

Figure 8-3. Section of river showing different controls at different water levels and a flood moving downstream

of a gauging station where the water level is regularly measured is given in Figure 8-3 which shows a longitudinal section of a stream. Downstream of the gauging station is usually some sort of fixed control which may be some local topography such as a rock ledge which means that for relatively small flows there is a relationship between the head over the control and the discharge which passes. This will control the flow for small flows. For larger flows the effect of the fixed control is to ”drown out”, to become unimportant, and for some other part of the stream to control the flow, such as the larger river downstream shown as a distant control in the figure, or even, if the downstream channel length is long enough before encountering another local control, the section of channel downstream will itself become the control, where the control is due to friction in the channel, giving a relationship between the slope in the channel, the channel geometry and roughness and the flow. There may be more controls too, but however many there are, if the channel were stable, and the flow steady (i.e. not changing with time anywhere in the system) there would be a unique relationship between stage and discharge, however complicated this might be due to various controls. In practice, the natures of the controls are usually unknown. Previously we have exploited the fact that a good approximation to the momentum equation for low Froude number flows is simply equation (6.5) or the equivalent (6.6), where it is likely that we can express conveyance K as a function of the local elevation at a gauging station:

$$Q = K(\eta)\sqrt{-\frac{\partial \eta}{\partial x}}.$$

We have seen that this is quite a faithful reflection of the actual mechanics of the situation, and shows that were we to measure the surface elevation η and its gradient $\partial\eta/\partial x$ at a point on a river, then provided we knew $K(\eta)$ we would have a quite accurate expression for the discharge. This implies that at a particular gauging station the computed discharge is a function of both η and $\partial\eta/\partial x$. However, the tradition in river engineering has been to assume that the surface slope varies little from mean bed slope. That is, to assume that

$$Q \approx K(\eta)\sqrt{S_0},$$

so that discharge Q is inferred to be a unique function of the *stage*, a term for the surface elevation, whether relative to a national datum or a local datum, and to ignore effects of surface gradient. Effectively, this has been to assume that the surface gradient is a constant for a given stage, whether at the front of a flood wave, or at the rear, or any other time.

8.3 Rating curves

In practice, at a gauging station the relationship between stage η and discharge Q has been continuously measured and approximated for a long time, as it can change with time due to downstream conditions changing or the bed moving, for example. At irregular intervals, such as quarterly, hydrographers measure the stage and measure the flow accurately by measuring the water velocity at closely-spaced intervals over the cross-section and using numerical integration to obtain the discharge. This provides a single point on the *Rating Curve*, a plot of discharge horizontally against stage vertically. From a number of such points, the rating curve is built up. Subsequently it is assumed that for any stage reading, the routine periodic measurement, the corresponding discharge, can be read off. This is what happens then, at intervals such as hourly or daily, the stage is read and telemetered to a central data management authority. From the rating curve for that stage, the corresponding discharge can be calculated. The figure below shows the current rating curve for the Ovens River at Wangaratta, where flow measurements have been made since 1891. There are a couple of difficulties with such a curve, including reading results off for small flows, where the curve is locally vertical, and for high flows where it is almost horizontal. A traditional way of overcoming the difficulty of representing rating curves over a large range has been to use log-log axes. However, this has no physical basis and has a number of practical difficulties, although it has been recommended by International and Australian Standards.Hydraulic theory can help here, for it can be used to show that the stage-discharge relationship will tend to show stage varying approximately like $\eta \sim Q^{1/2}$, for both cases:

1. Flow across a U-shaped (parabolic) weir, the approximate situation for *low flow* at a gauging station, when a local control such as a rock ledge controls the flow, and
2. Uniform flow down a U-shaped (parabolic) waterway for *large flows*, when the local control is washed out and the waterway acts more like a uniform flow governed by Manning's law.

In these cases, both parts of the relationship would plot as (different) straight lines on $\left(\sqrt{Q},\eta\right)$ axes. Here we plot the results from the above figure on such a square root scale for the discharge, and we see that indeed at both small and large flows the rating curve is a straight line. This means that simpler procedures of numerical approximation and interpolation could be used. Sometimes results have to be taken by extrapolating the curve. If this has to be done, then linear extrapolation on the $\left(\sqrt{Q},\eta\right)$ axes might be reasonable, but it is still a procedure to be followed with great caution, as the actual geometry

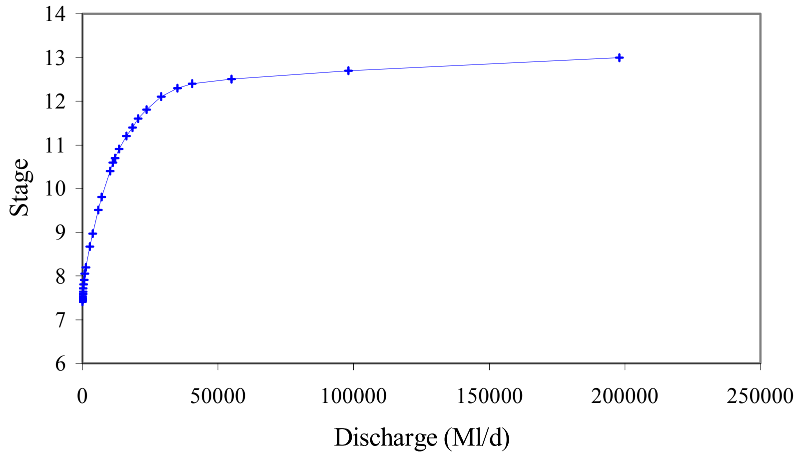

Figure 8-4. Rating curve using natural (Q, η) axes

for above-bank flows can vary a lot.In all of this we have not considered what actually happens to a

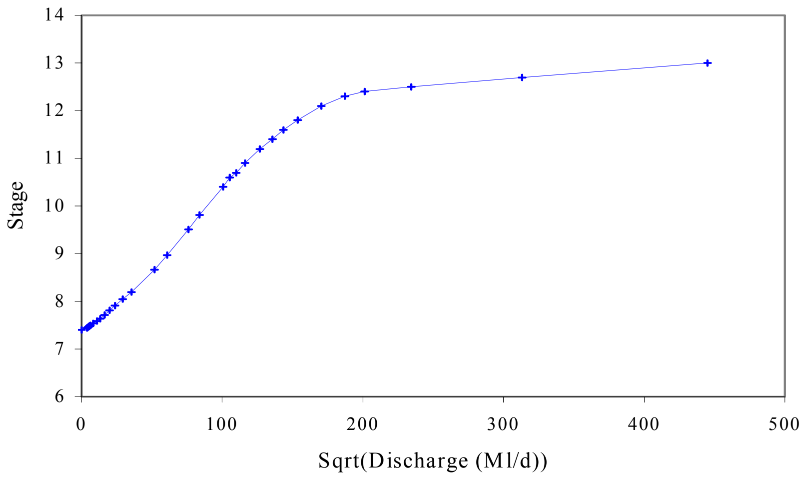

Figure 8-5. A rating curve using $(\sqrt{Q}, \eta)$ axes

rating curve if the depth gradient $\partial h/\partial x$ is significant. Usually it is not important, but for a sufficiently rapidly rising and falling flood we can imagine that it might be. Consider the front of a flood wave where the depth increases towards the top of the flood. In this case $\partial h/\partial x$ will be negative, and equation (6.6)

$$Q = K\sqrt{S_0 - \frac{\partial h}{\partial x}},$$

shows that the discharge will be larger than if we had neglected this slope. Similarly, at the rear of the flood wave the depth gradient will be positive, and the flow will be rather less. In this situation the trajectory of the flood actually forms a loop on the Stage-discharge diagram. There are some ways around this. One would be to recognise that in general, the discharge is not just a function of stage but also depends on the surface slope. Hydraulics has told us that it might be better always to measure slope as well as stage by measuring the stage at two points, and to develop a Stage-Conveyance relationship rather than a Stage-Discharge relationship. This, however, lies in the future. Some recent research at the Co-operative Research Centre for Catchment Hydrology has investigated some of these aspects and produced some solutions.

9. Loose-boundary hydraulics

Most rivers and canals have beds of soil, of a more-or-less erodible nature. Water flowing in such streams has the ability to scour the bed, to carry particles which are heavier than the water, and to deposit material, hence changing the bed topography. This phenomenon is of great economic and ecological importance, for example in predicting the scouring around and potential collapse of bridges, weirs, channel banks *etc*., estimating the rate of siltation of reservoirs, predicting the possible form changes of rivers with a threat to aquatic life *etc.*.

9.1 Sediment transport

The grains forming the boundary of an alluvial stream have a finite weight and finite resisting ability, including cohesion and coefficient of friction. They can be brought into motion if the forces due to fluid motion acting on a sediment particle are greater than the resisting forces. Often this is expressed in terms of disturbing and resisting stresses on the bed of the stream. If the shear stress τ acting at a point on the flow boundary is greater than a certain critical value τ_{cr} then grains will be removed from that region, and the bed is said to *scour* there. We introduce the concept of a *relative tractive force* at a point, τ/τ_{cr}. If this is slightly greater than 1, only the grains forming the uppermost layer of the flow boundary can be detached and transported. If τ/τ_{cr} is greater than 1, but less than a certain amount, then grains are transported by deterministic jumps in the neighbourhood of the bed. This mode of grain transport is referred to as *bed-load*. If the ratio is large, then grains will be entrained into the flow and will be carried downstream by turbulence. This transport mechanism is known as *suspended-load*. The total transport rate is the sum of the two. The simultaneous motion of the transporting fluid and the transported sediment is a form of *two-phase flow*. We can write all the variables which should dominate the problem of the removal and transport of particles:

ρ	Density of water	ML^{-3}
ρ_s	Density of solid particles	ML^{-3}
ν	Kinematic viscosity of water	$\mathrm{L}^2\mathrm{T}^{-1}$
ϕ	Diameter of grain	L
g	Gravitational acceleration	LT^{-2}
h	Depth of flow	L
τ	Shear stress of water on bed	$\mathrm{ML}^{-1}\mathrm{T}^{-2}$

As we have 7 such quantities and 3 fundamental dimensions involved, there are 4 dimensionless numbers which can characterise the problem. In fact, it is convenient to replace g by $g' = g\,(\rho_s/\rho - 1)$, the apparent submerged gravitational acceleration of the particles, and to replace τ by the shear velocity $u_* = \sqrt{\tau/\rho}$. Convenient dimensionless variables, partly found from physical considerations, which occur are

$$\Theta = \frac{u_*^2}{g'\phi} = \frac{\tau}{(\rho_s - \rho)\, g\phi}, \quad \text{roughly the ratio of the shear force on a particle to its submerged weight}$$

$$R_* = \frac{u_*\phi}{\nu}, \quad \text{roughly the ratio of fluid inertia forces to viscous forces on the grain}$$

$$G = \frac{\rho_s}{\rho}, \quad \text{the specific gravity of the bed material, and}$$

$$\frac{\phi}{h}, \quad \text{the ratio of grain size to water depth}$$

Two important quantities here are R_* which is the *grain Reynolds number*, and Θ the *Shields parameter*, which can be thought of as a dimensionless stress.

9.2 Incipient motion

In the 1930s Shields conducted a number of experiments in Berlin and found that there was a narrow band of demarcation between motion and no motion of bed particles, corresponding to incipient motion. He represented these on a figure of Θ versus R_*. A slight problem with this is that the fluid velocity (in the form of shear velocity) occurs in both quantities. It is more reasonable to introduce the dimensionless grain size (see p7 of Yalin & Ferreira da Silva 2001):

$$\delta = \left(\frac{R_*^2}{\Theta}\right)^{1/3} = \phi\left(\frac{g'}{\nu^2}\right)^{1/3}.$$

Here we consider what δ means. If we take a common value of $G = 2.65$, plus $g = 9.8\,\mathrm{m\,s^{-2}}$, $\nu = 10^{-6}\,\mathrm{m^2\,s^{-1}}$(for 20°C), then we obtain $\delta \approx \phi \times 25000$ in units of metres. If ϕ is specified in terms of millimetres then we have $\delta \approx 25\,\phi$, and so for a range of particle sizes we have

δ	0.1	1	10	100	1000
ϕ (mm)	0.004	0.04	0.4	4	40

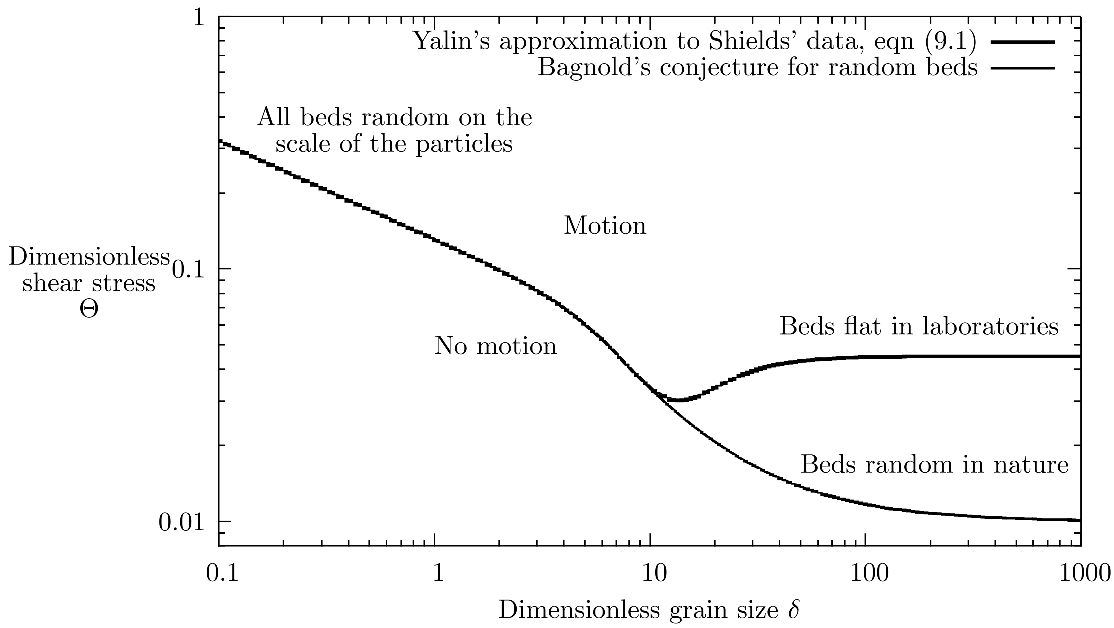

Figure 9-1. Incipient motion diagram

Figure 9-1 shows a representation of Shields' results, using δ for the abcissa. Instead of the experimental results we use a formula by Yalin which is an approximation to the results for incipient motion, giving the critical value Θ_{cr}:

$$\Theta_{\mathrm{cr}} = 0.13\,\delta^{-0.392}\,e^{-0.015\,\delta^2} + 0.045\,\left(1 - e^{-0.068\,\delta}\right). \tag{9.1}$$

Above the line, for larger values of Θ (and hence larger velocities or smaller and lighter grains), particles will be entrained into the flow. Below the line, particles should be stable. For small particles there appears to be a linear relationship (on these log-log axes), while for large particles the critical shear stress, based on Shields' laboratory experiments, approaches a constant value of about 0.045. In between there is a dip in the curve, with a minimum at about $\phi = 14$, corresponding to a grain size of $0.35\,\mathrm{mm}$, about a fine sand.

Bagnold (personal communication) suggested that there was probably no fluid mechanical reason for that, but that there is an implicit scale effect in the diagram, an artificial geometric effect, and suggested that the Shields diagram has been widely misinterpreted. He suggested that for experiments with small particles, while the overall bed may have been flattened, individual small grains may sit on top of others and may project into the flow, so that the assemblage is random on a small scale. For large particles (gravel, boulders, *etc.*) in nature, they too are free to project into the flow, however in the experiments which determined the Shields diagram, the bed was made flat by levelling the tops of the large particles. Hence, there is an artificial scale effect, and if one were only to consider random beds of particles which are free to project into the flow above their immediate neighbours, while the bed might level on a scale much larger than the particles. Fenton & Abbott (1977) followed Bagnold's suggestion and examined the effect of protrusion of particles into the stream. Although they did not obtain definitive results, they were able to recommend that for large particles the value of Θ_{cr} was more like 0.01 than 0.045, which seems to be an important difference, the factor of $1/4$ requiring a fluid velocity for entrainment into the flow of randomly-placed particles to be about half that of the sheltered case. A curve representing Bagnold's hypothesis, as partly borne out by the experiments, is shown on Figure 9-1.

9.3 Turbulent flow in streams

Now we consider some simple relations to relate this to physical quantities. The shear velocity u_* is a very convenient quantity indeed. If we consider the steady uniform flow in a channel, then the component of gravity force down the channel on a slice of length Δx is $\rho g A\,\Delta x\,S_0$. However the shear force resisting the gravity force is $\tau \times P \times \Delta x$. Equating the two we obtain

$$\tau = \rho g \frac{A}{P}\, S_0.$$

In this work it is sensible only to consider the wide channel case, such that $A/P = h$, the depth, giving

$$\tau = \rho g h\, S_0,$$

or in terms of the shear velocity:

$$u_* = \sqrt{\frac{\tau}{\rho}} = \sqrt{g h\, S_0},$$

and so in terms of the dimensionless stress:

$$\Theta = \frac{u_*^2}{g'\phi} = \frac{g h\, S_0}{g'\phi} = \frac{S_0\, h}{(G-1)\,\phi}.$$

9.4 Dimensional similitude

In experiments with sediment transport, as in other areas of fluid mechanics, it is desirable to have the same dimensionless numbers governing both experimental and full-scale situations. In this case we would like the dimensionless particle size AND the dimensionless shear stress each to have the same values in both model and full scale. Using the subscript m for model and no subscript for the full scale situation, we then should have

$$\delta_m = \delta \quad \text{such that} \quad \phi_m \left(\frac{g'_m}{\nu^2}\right)^{1/3} = \phi \left(\frac{g'}{\nu^2}\right)^{1/3},$$

but as gravitational acceleration g and viscosity ν are the same in each, we can write

$$\phi_m\,(G_m - 1)^{1/3} = \phi\,(G-1)^{1/3}.$$

Also we require the same dimensionless shear stress:

$$\frac{S_{0m}\, h_m}{(G_m - 1)\,\phi_m} = \frac{S_0\, h}{(G-1)\,\phi}.$$

In practice it is difficult to be able to satisfy all the dimensionless numbers.

9.5 Bed-load rate of transport – Bagnold's formula

The volumetric rate of transport q_{sb} per unit width is given by

$$q_{sb} = \frac{\beta u_b (\tau - \tau_{\mathrm{cr}})}{(\rho_s - \rho)\, g},$$

where β is a function of δ, and u_b is the flow velocity in the vicinity of the bed. In the case of a rough turbulent flow, $\beta \approx 0.5$. This formula is preferred, as it is simple, as accurate as any, and reflects the meaning of the bed-load rate.

9.6 Bedforms

In most practical situations, sediments behave as non-cohesive materials, and the fluid flow can distort the bed into various shapes. The interaction process is complex. At low velocities the bed does not move. With increasing flow velocity the inception of movement occurs. The basic bed forms encountered are *ripples* (usually of heights less than $0.1\,\mathrm{m}$), *dunes*, *flat bed*, *standing waves*, and *antidunes*. At high velocities chutes and step-pools may form. Typical bed forms are summarised in Figure 9-2 below.

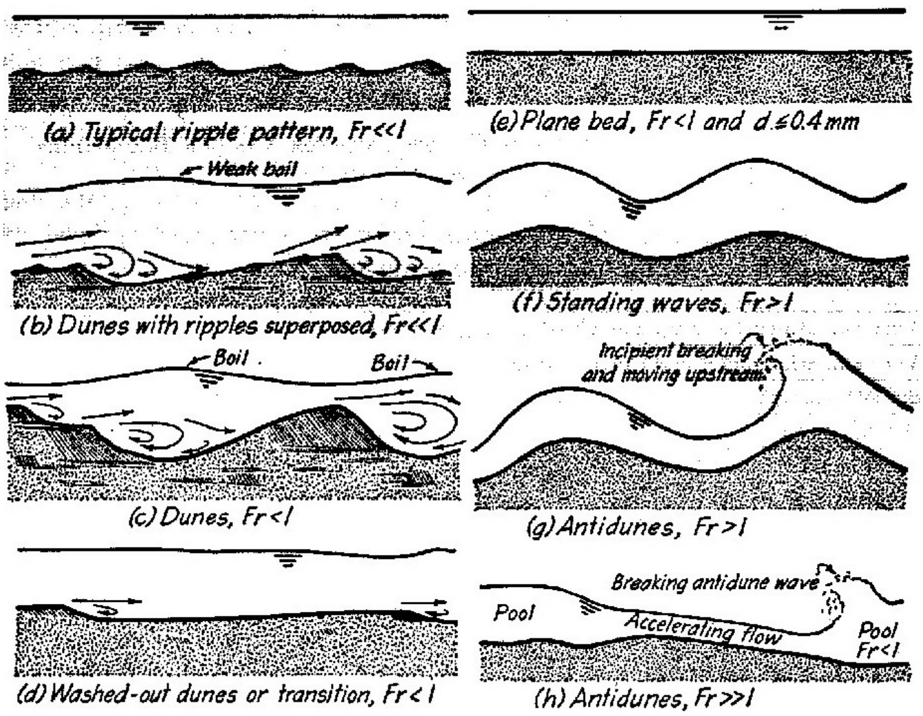

Figure 9-2. Bedforms and the Froude numbers at which they occur (after Richardson and Simons)