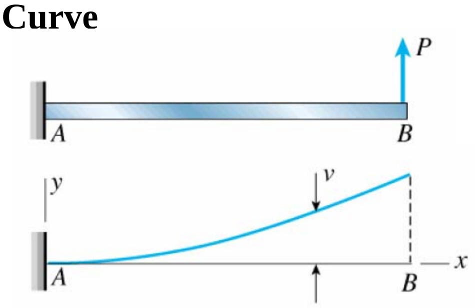
Chapter 9　Deflections of Beams

9.1 Introduction

in this chapter, we describe methods for determining the equation of the deflection curve of beams and finding deflection and slope at specific points along the axis of the beam

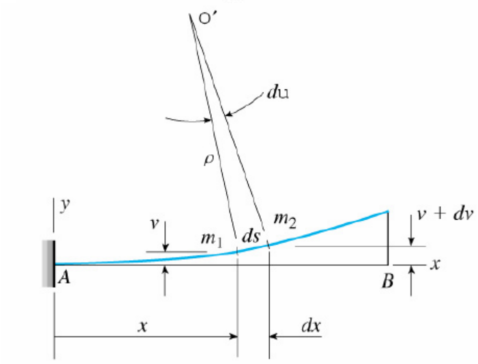
9.2 Differential Equations of the Deflection Curve

consider a cantilever beam with a concentrated load acting upward at the free end



the deflection v is the displacement in the y direction

the angle of rotation θ of the axis (also called slope) is the angle between the x axis and the tangent to the deflection curve



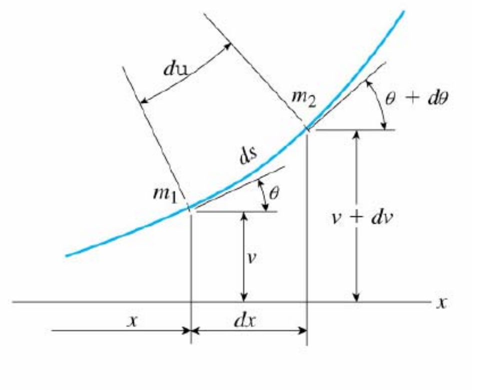
(a)

point m_1 is located at distance x

point m_2 is located at distance $x + dx$

slope at m_1 is θ

slope at m_2 is $\theta + d\theta$



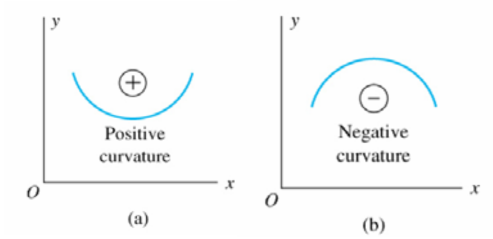
(b)

denote O' the center of curvature and ρ the radius of curvature, then

$$\rho \, d\theta \;=\; ds$$

and the curvature κ is

$$\kappa = \frac{1}{\rho} = \frac{d\theta}{ds}$$



the sign convention is pictured in figure

slope of the deflection curve

$$\frac{dv}{dx} = \tan\theta \qquad \text{or} \qquad \theta = \tan^{-1}\frac{dv}{dx}$$

for θ small $\quad ds \simeq dx \quad \cos\theta \simeq 1 \quad \tan\theta \simeq \theta$, then

$$\kappa = \frac{1}{\rho} = \frac{d\theta}{dx} \qquad \text{and} \qquad \theta = \frac{dv}{dx}$$

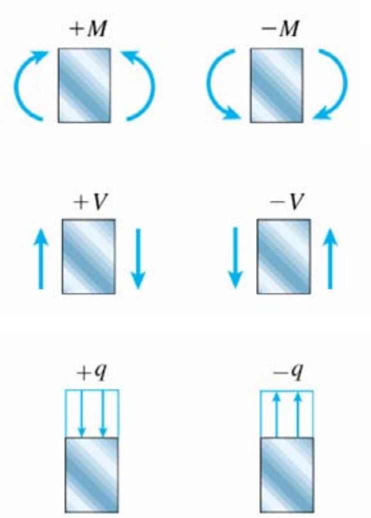
$$\kappa = \frac{1}{\rho} = \frac{d\theta}{dx} = \frac{d^2v}{dx^2}$$

if the materials of the beam is linear elastic

$$\kappa = \frac{1}{\rho} = \frac{M}{EI} \qquad \text{[chapter 5]}$$

then the differential equation of the deflection curve is obtained

$$\boldsymbol{\frac{d\theta}{dx} = \frac{d^2v}{dx^2} = \frac{M}{EI}}$$



it can be integrated to find θ and v

$$\because \qquad \frac{dM}{dx} = V \qquad \frac{dV}{dx} = -q$$

then $$\frac{d^3v}{dx^3} = \frac{V}{EI} \qquad \frac{d^4v}{dx^4} = -\frac{q}{EI}$$

sign conventions for M, V and q are shown

the above equations can be written in a simple form

$$EIv'' = M \qquad EIv''' = V \qquad EIv'''' = -q$$

this equations are valid only when Hooke's law applies and when the slope and the deflection are very small

for nonprismatic beam [$I = I(x)$], the equations are

$$EI_x \frac{d^2v}{dx^2} = M$$

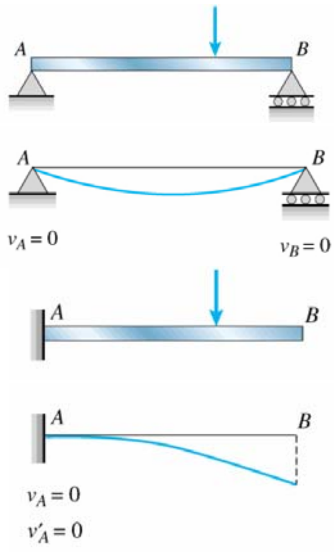
$$\frac{d}{dx}(EI_x \frac{d^2v}{dx^2}) = \frac{dM}{dx} = V$$

$$\frac{d^2}{dx^2}(EI_x \frac{d^2v}{dx^2}) = \frac{dV}{dx} = -q$$

the exact expression for curvature can be derived

$$\kappa = \frac{1}{\rho} = \frac{v''}{[1 + (v')^2]^{3/2}}$$

9.3 Deflections by Integration of the Bending-Moment Equation



substitute the expression of $M(x)$ into the deflection equation then integrating to satisfy

(i) boundary conditions

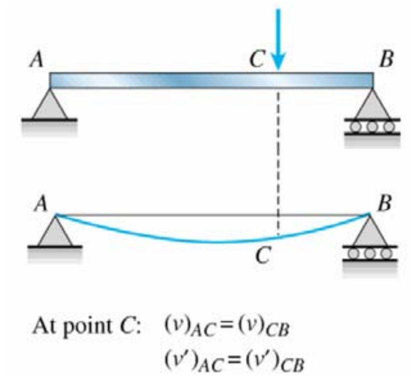
(ii) continuity conditions

(iii) symmetry conditions

to obtain the slope θ and the

deflection v of the beam

this method is called method of successive integration

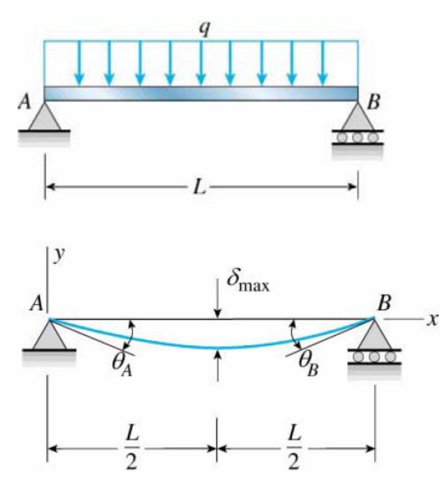


Example 9-1

determine the deflection of beam AB supporting a uniform load of intensity q

also determine δ_{max} and θ_A, θ_B

flexural rigidity of the beam is EI

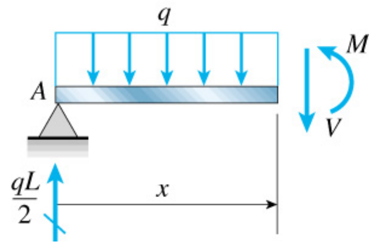


(b)

bending moment in the beam is

$$M = \frac{qLx}{2} - \frac{q\,x^2}{2}$$

differential equation of the deflection curve



$$EI\,v'' = \frac{qLx}{2} - \frac{q\,x^2}{2}$$

Then

$$EI\,v' = \frac{qLx^2}{4} - \frac{q\,x^3}{6} + C_1$$

$\because$ the beam is symmetry, $\therefore$ $\theta = v' = 0$ at $x = L / 2$

$$0 = \frac{qL(L/2)^2}{4} - \frac{q\,(L/2)^3}{6} + C_1$$

then $C_1 = q L^3 / 24$

the equation of slope is

$$v' = - \frac{q}{24 EI} (L^3 - 6 L x^2 + 4 x^3)$$

integrating again, it is obtained

$$v = - \frac{q}{24 EI} (L^3 x - 2 L x^3 + x^4) + C_2$$

boundary condition : $v = 0$ at $x = 0$

thus we have $C_2 = 0$

then the equation of deflection is

$$v = - \frac{q}{24 EI} (L^3 x - 2 L x^3 + x^4)$$

maximum deflection δ_{max} occurs at center $(x = L/2)$

$$\delta_{max} = - v(\frac{L}{2}) = \frac{5 q L^4}{384 EI} \qquad (\downarrow)$$

the maximum angle of rotation occurs at the supports of the beam

$$\theta_A = v'(0) = - \frac{q L^3}{24 EI} \qquad (\circlearrowright)$$

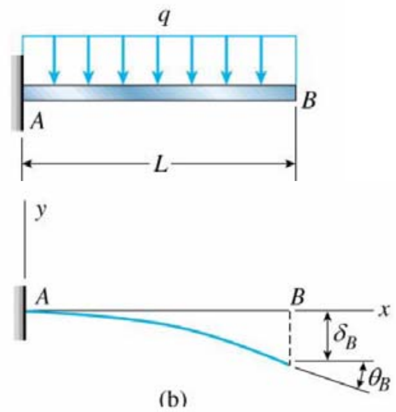
and $$\theta_B = v'(L) = \frac{q L^3}{24 EI} \qquad (\circlearrowleft)$$

Example 9-2

determine the equation of deflection curve for a cantilever beam AB subjected to a uniform load of intensity q

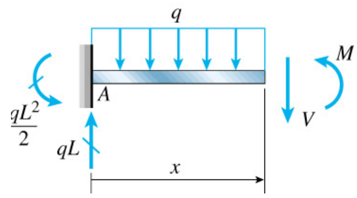
also determine θ_B and δ_B at the free end

flexural rigidity of the beam is EI



bending moment in the beam

$$M = -\frac{q L^2}{2} + q L x - \frac{q x^2}{2}$$



$$EIv'' = -\frac{q L^2}{2} + q L x - \frac{q x^2}{2}$$

$$EIy' = -\frac{qL^2x}{2} + \frac{qLx^2}{2} - \frac{q x^3}{6} + C_1$$

boundary condition $v' = \theta = 0$ at $x = 0$

$$C_1 = 0$$

$$v' = -\frac{qx}{6EI}(3 L^2 - 3 L x + x^2)$$

integrating again to obtain the deflection curve

$$v = -\frac{qx^2}{24EI}(6 L^2 - 4 L x + x^2) + C_2$$

boundary condition $v = 0$ at $x = 0$

$$C_2 = 0$$

then

$$v = -\frac{qx^2}{24EI}(6L^2 - 4Lx + x^2)$$

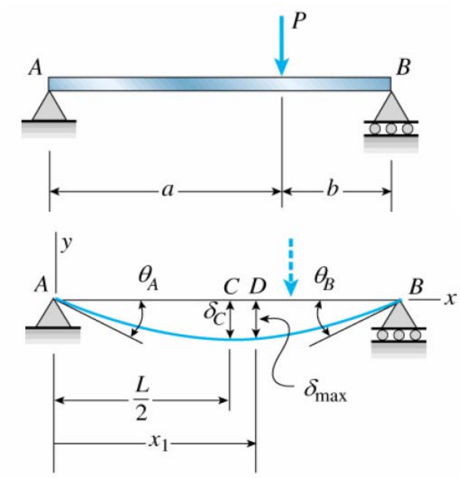
$$\theta_{max} = \theta_B = v'(L) = -\frac{qL^3}{6EI} \quad (\circlearrowright)$$

$$\delta_{max} = -\delta_B = -v(L) = \frac{qL^4}{8EI} \quad (\downarrow)$$

Example 9-4

determine the equation of deflection curve, θ_A, θ_B, δ_{max} and δ_C

flexural rigidity of the beam is EI



(b)

bending moments of the beam

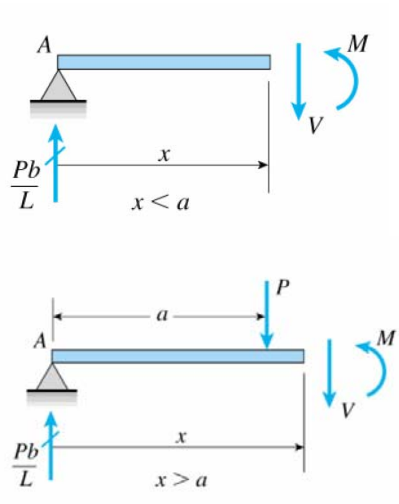
$$M = \frac{Pbx}{L} \qquad (0 \leqq x \leqq a)$$

$$M = \frac{Pbx}{L} - P(x-a) \quad (a \leqq x \leqq L)$$

differential equations of the deflection curve

$$EIv'' = \frac{Pbx}{L} \quad (0 \leqq x \leqq a)$$

$$EIv'' = \frac{Pbx}{L} - P(x-a) \quad (a \leqq x \leqq L)$$



(b)

integrating to obtain

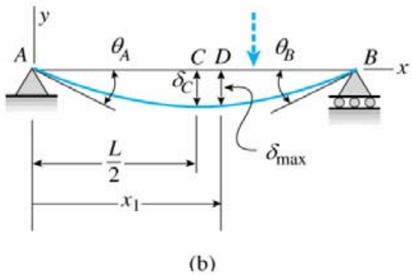
$$EIv' = \frac{Pbx^2}{2L} + C_1 \quad (0 \leqq x \leqq a)$$

$$EIv' = \frac{Pbx^2}{2L} - \frac{P(x-a)^2}{2} + C_2 \quad (a \leqq x \leqq L)$$

2nd integration to obtain

$$EIv = \frac{Pbx^3}{6L} + C_1 x + C_3 \quad (0 \leqq x \leqq a)$$

$$EIv = \frac{Pbx^3}{6L} - \frac{P(x-a)^3}{6} + C_2 x + C_4 \quad (a \leqq x \leqq L)$$



boundary conditions

(i) $v(0) = 0$ (ii) $y(L) = 0$

continuity conditions

(iii) $v'(a^-) = v'(a^+)$ (iv) $v(a^-) = v(a^+)$

(i) $v(0) = 0$ => $C_3 = 0$

(ii) $v(L) = 0$ => $\frac{PbL^3}{6} - \frac{Pb^3}{6} + C_2 L + C_4 = 0$

(iii) $v'(a^-) = v'(a^+)$ => $\frac{Pba^2}{2L} + C_1 = \frac{Pba^2}{2L} + C_2$

$C_1 = C_2$

(iv) $v(a^-) = v(a^+)$ => $\frac{Pba^3}{6L} + C_1 a + C_3 = \frac{Pba^3}{6L} + C_2 a + C_4$

$C_3 = C_4$

then we have

$$C_1 = C_2 = -\frac{Pb\,(L^2 - b^2)}{6L}$$

$$C_3 = C_4 = 0$$

thus the equations of slope and deflection are

$$v' = -\frac{Pb}{6LEI}(L^2 - b^2 - 3x^2) \qquad (0 \leqq x \leqq a)$$

$$v' = -\frac{Pb}{6LEI}(L^2 - b^2 - 3x^2) - \frac{P(x - a)^2}{2EI} \qquad (a \leqq x \leqq L)$$

$$v = -\frac{Pbx}{6LEI}(L^2 - b^2 - x^2) \qquad (0 \leqq x \leqq a)$$

$$v = -\frac{Pbx}{6LEI}(L^2 - b^2 - x^2) - \frac{P(x - a)^3}{6EI} \qquad (a \leqq x \leqq L)$$

angles of rotation at supports

$$\theta_A = v'(0) = -\frac{Pab(L + b)}{6LEI} \qquad (\circlearrowright)$$

$$\theta_B = v'(L) = \frac{Pab(L + a)}{6LEI} \qquad (\circlearrowleft)$$

$\because$ θ_A is function of a (or b), to find $(\theta_A)_{max}$, set $d\theta_A / db = 0$

$$\theta_A = -\frac{Pb(L^2 - b^2)}{6LEI}$$

$$d\theta_A / db = 0 \quad \Rightarrow \quad L^2 - 3b^2 = 0 \quad \Rightarrow \quad b = L / \sqrt{3}$$

$$(\theta_A)_{max} = -\frac{PL^2\sqrt{3}}{27\ EI}$$

for maximum δ occurs at x_1, if $a > b$, $x_1 < a$

$$\frac{dv}{dx} = 0 \quad \Rightarrow \quad x_1 = \frac{L^2 - b^2}{3} \qquad (a \geqq b)$$

$$\delta_{max} = -v(x_1) = \frac{Pb(L^2 - b^2)^{3/2}}{9\sqrt{3}\,LEI} \qquad (\downarrow)$$

at $x = L/2$ $$\delta_C = -v(L/2) = \frac{Pb(3L^2 - 4b^2)}{48\ EI} \qquad (\downarrow)$$

$\because$ the maximum deflection always occurs near the midpoint, $\therefore$ δ_C gives a good approximation of the δ_{max}

in most case, the error is less than 3%

an important special case is $a = b = L/2$

$$v' = \frac{P}{16EI}(L^2 - 4x^2) \qquad (0 \leqq x \leqq L/2)$$

$$v = \frac{P}{48EI}(3L^2 - 4x^2) \qquad (0 \leqq x \leqq L/2)$$

v' and v are symmetric with respect to $x = L/2$

$$\theta_A = \theta_B = \frac{PL^2}{16EI}$$

$$\delta_{max} = \delta_C = \frac{PL^3}{48EI}$$

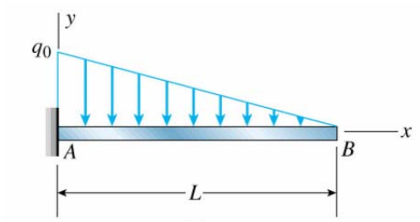
9.4 Deflections by Integration of Shear-Force and Load Equations

the procedure is similar to that for the bending moment equation except that more integrations are required

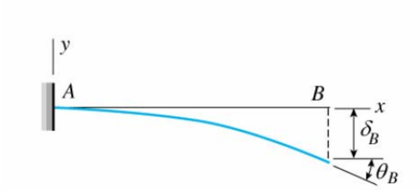
if we begin from the load equation, which is of fourth order, four integrations are needed

Example 9-4

determine the equation of deflection curve for the cantilever beam AB supporting a triangularly distributed load of maximum intensity q_0



(a)



(b)

also determine δ_B and θ_B

flexural rigidity of the beam is EI

$$q = \frac{q_0 (L - x)}{L}$$

$$EIv'''' = - q = - \frac{q_0 (L - x)}{L}$$

the first integration gives

$$EIv''' = - \frac{q_0 (L - x)^2}{2 L} + C_1$$

$$\because \quad v'''(L) = V = 0 \quad => C_1 = 0$$

thus $\quad EIv''' = - \dfrac{q_0 (L - x)^2}{2 L}$

2nd integration

$$EIv'' = -\frac{q_0 (L - x)^3}{6L} + C_2$$

$$\because \quad v''(L) = M = 0 \quad => \quad C_2 = 0$$

thus $\quad EIv'' = -\dfrac{q_0 (L - x)^3}{6L}$

3rd and 4th integrations to obtain the slope and deflection

$$EIv' = -\frac{q_0 (L - x)^4}{24L} + C_3$$

$$EIv = -\frac{q_0 (L - x)^5}{120L} + C_3 x + C_4$$

boundary conditions : $v'(0) = v(0) = 0$

the constants C_3 and C_4 can be obtained

$$C_3 = -\frac{q_0 L^3}{24} \qquad C_4 = \frac{q_0 L^4}{120}$$

then the slope and deflection of the beam are

$$v' = -\frac{q_0 x}{24LEI}(4L^3 - 6L^2 x + 4Lx^2 - x^3)$$

$$v = -\frac{q_0 x^2}{120LEI}(10L^3 - 10L^2 x + 5Lx^2 - x^3)$$

$$\theta_B = v'(L) = -\frac{q_0 L^3}{24\,EI} \qquad (↻)$$

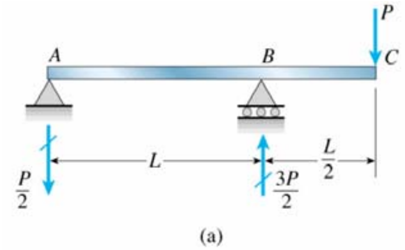
$$\delta_B = -v(L) = \frac{q_0 L^4}{30\,EI} \qquad (\downarrow)$$

Example 9-5

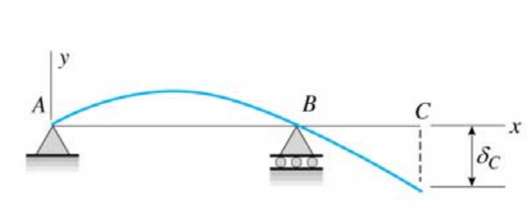
an overhanging beam ABC with a concentrated load P applied at the end

determine the equation of deflection curve and the deflection δ_C at the end

flexural rigidity of the beam is EI



(a)



(b)

the shear forces in parts AB and BC are

$$V = -\frac{P}{2} \qquad (0 < x < L)$$

$$V = P \qquad (L < x < \frac{3L}{2})$$

the third order differential equations are

$$EIv''' = -\frac{P}{2} \qquad (0 < x < L)$$

$$EIv''' = P \qquad (L < x < \frac{3L}{2})$$

bending moment in the beam can be obtained by integration

$$M = EIv'' = -\frac{Px}{2} + C_1 \qquad (0 \leqq x \leqq L)$$

$$M = EIv'' = Px + C_2 \qquad (L \leqq x \leqq \frac{3L}{2})$$

boundary conditions : $v''(0) = v''(3L/2) = 0$

we get $C_1 = 0 \qquad C_2 = -\dfrac{3PL}{2}$

therefore the bending moment equations are

$$M = EIv'' = -\frac{Px}{2} \qquad (0 \leqq x \leqq L)$$

$$M = EIv'' = -\frac{P(3L - 2x)}{2} \qquad (L \leqq x \leqq \frac{3L}{2})$$

2nd integration to obtain the slope of the beam

$$EIv' = -\frac{Px^2}{4} + C_3 \qquad (0 \leqq x \leqq L)$$

$$EIv' = -\frac{Px(3L - x)}{2} + C_4 \quad (L \leqq x \leqq \frac{3L}{2})$$

continuity condition : $v'(L^-) = v'(L^+)$

$$-\frac{PL^2}{4} + C_3 = -PL^2 + C_4$$

then $C_4 = C_3 + \dfrac{3PL^2}{4}$

the 3rd integration gives

$$EIv = -\frac{Px^3}{12} + C_3\,x + C_5 \qquad (0 \leqq x \leqq L)$$

$$EIv = -\frac{Px^2(9L - 2x)}{12} + C_4\,x + C_6 \quad (L \leqq x \leqq \frac{3L}{2})$$

boundary conditions : $v(0) = v(L^-) = 0$

we obtain

$$C_5 = 0 \qquad C_3 = \frac{PL^2}{12}$$

and then $C_4 = \dfrac{5PL^2}{6}$

the last boundary condition : $v(L^+) = 0$

then $C_6 = -\dfrac{PL^3}{4}$

the deflection equations are obtained

$$v = \frac{Px}{12EI}(L^2 - x^2) \qquad (0 \leqq x \leqq L)$$

$$v = -\frac{P}{12EI}(3L^3 - 10L^2x + 9Lx^2 - 2x^3) \qquad (L \leqq x \leqq \frac{3L}{2})$$

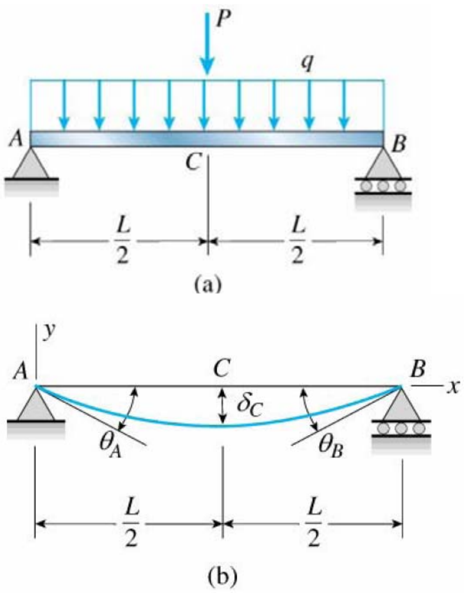
$$= -\frac{P}{12EI}(3L - x)(L - x)(L - 2x)$$

deflection at C is

$$\delta_C = -v(\frac{3L}{2}) = \frac{PL^3}{8EI} \quad (\downarrow)$$

9.5 Method of Superposition

the slope and deflection of beam caused by several different loads acting simultaneously can be found by superimposing the slopes and deflections caused by the loads acting separately consider a simply beam supports two loads : (1) uniform load of intensity q and (2) a concentrated load P



the slope and deflection due to load ***1*** are

$$(\delta_C)_1 = \frac{5qL^4}{384EI}$$

$$(\theta_A)_1 = (\theta_B)_1 = \frac{qL^3}{24EI}$$

the slope and deflection due to load ***2*** are

$$(\delta_C)_2 = \frac{PL^3}{48EI} \qquad (\theta_A)_2 = (\theta_B)_2 = \frac{PL^2}{16EI}$$

therefore the deflection and slope due to the combined loading are

$$\delta_C = (\delta_C)_1 + (\delta_C)_2 = \frac{5qL^4}{384EI} + \frac{PL^3}{48EI}$$

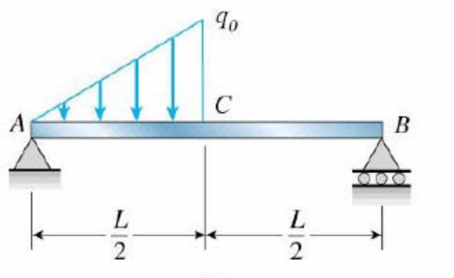
$$\theta_A = \theta_B = (\theta_A)_1 + (\delta_A)_2 = \frac{qL^3}{24EI} + \frac{PL^2}{16EI}$$

tables of beam deflection for simply and cantilever beams are given in Appendix G

superposition method may also be used for distributed loading

consider a simple beam ACB with a triangular load acting on the left-hand half the deflection of midpoint due to a concentrated load is obtained [table G-2]

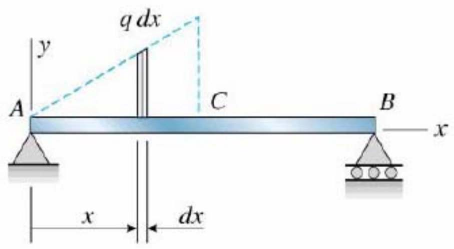
$$\delta_C = \frac{Pa}{48EI}(3L^2 - 4a^2)$$



(a)

substitute $q\,dx$ for P and x for a

$$d\delta_C = \frac{(qdx)\,x}{48EI}(3L^2 - 4x^2)$$



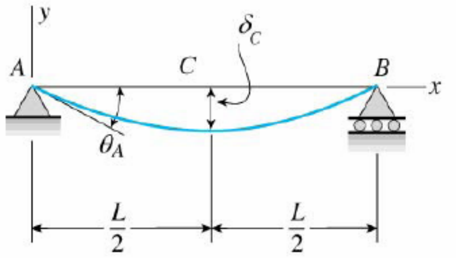
(b)

the intensity of the distributed load is

$$q = \frac{2q_0x}{L}$$

then δ_C due to the concentrated load q

$$d\delta_C = \frac{q_0\,x^2}{24LEI}(3L^2 - 4x^2)\,dx$$



(c)

thus δ_C due to the entire triangular load is

$$\delta_C = \int_0^{L/2} \frac{q_0\,x^2}{24LEI}(3L^2 - 4x^2)\,dx = \frac{q_0L^4}{240EI}$$

similarly, the slope θ_A due to P acting on a distance a from left end is

$$d\theta_A = \frac{Pab(L + b)}{6LEI}$$

replacing P with $2q_0x\,dx/L$, a with x, and b with $(L - x)$

$$d\theta_A = \frac{2q_0x^2(L - x)(L + L - x)}{6L^2EI} dx = \frac{q_0}{3L^2EI} (L - x)(2L - x) x^2 dx$$

thus the slope at A throughout the region of the load is

$$\theta_A = \int_0^{L/2} \frac{q_0}{3L^2EI} (L - x)(2L - x) x^2 dx = \frac{41q_0L^3}{2880EI}$$

the principle of superposition is valid under the following conditions

(1) Hooke's law holds for the material

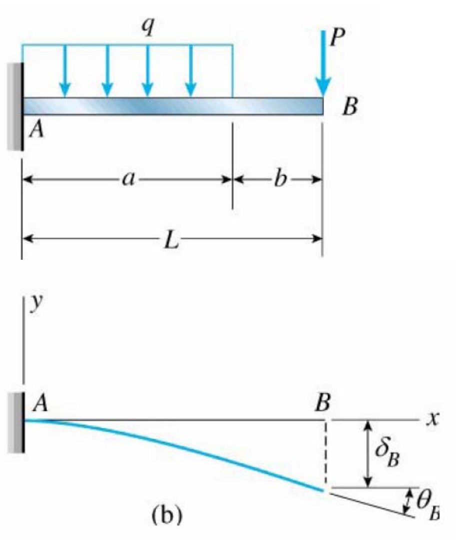
(2) the deflections and rotations are small

(3) the presence of the deflection does not alter the actions of applied loads

these requirements ensure that the differential equations of the deflection curve are linear

Example 9-6

a cantilever beam AB supports a uniform load q and a concentrated load P as shown



determine δ_B and θ_B

EI = constant

from Appendix G :

for uniform load q

$$(\delta_B)_1 = \frac{qa^3}{24EI}(4L - a) \qquad (\theta_B)_1 = \frac{qa^3}{6EI}$$

for the concentrated load P

$$(\delta_B)_2 = \frac{PL^3}{3EI} \qquad (\theta_B)_2 = \frac{PL^2}{2EI}$$

then the deflection and slope due to combined loading are

$$\delta_B = (\delta_B)_1 + (\delta_B)_2 = \frac{qa^3}{24EI}(4L - a) + \frac{PL^3}{3EI}$$

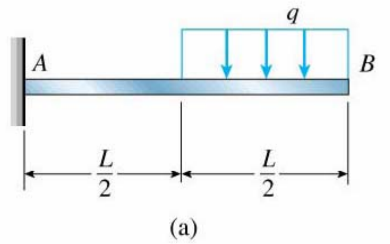
$$\theta_B = (\theta_B)_1 + (\theta_B)_2 = \frac{qa^3}{6EI} + \frac{PL^2}{2EI}$$

Example 9-7

a cantilever beam AB with uniform load q acting on the right-half

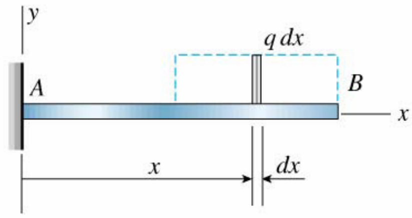
determine δ_B and θ_B

EI = constant



(a)

consider an element of load has magnitude $q\ dx$ and is located at distance x from the support



(b)

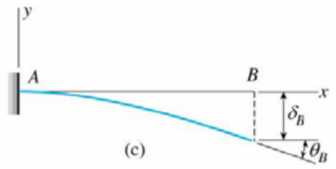
from Appendix G, table G-1, case 5

by replacing P with $q\ dx$ and a with x

$$d\delta_B = \frac{(qdx)(x^2)(3L-x)}{6EI} \qquad d\theta_B = \frac{(qdx)(x^2)}{2EI}$$

by integrating over the loaded region

$$\delta_B = \int_{L/2}^{L} \frac{qx^2(3L-x)}{6EI}\,dx = \frac{41qL^4}{384EI}$$



(c)

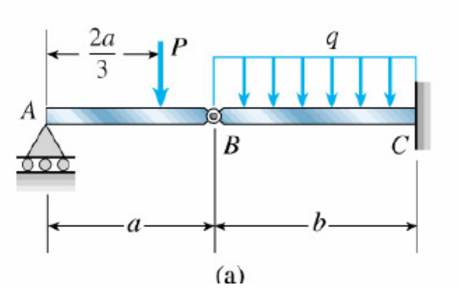
$$\theta_B = \int_{L/2}^{L} \frac{qx^2}{2EI}\,dx = \frac{7qL^3}{48EI}$$

Example 9-8

a compound beam ABC supports a concentrated load P and an uniform load q as shown

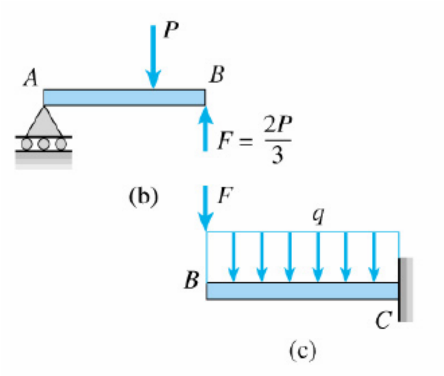
determine δ_B and θ_A

EI = constant



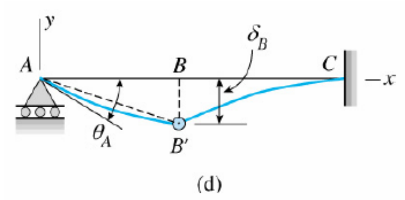
we may consider the beam as composed of two parts : (1) simple beam AB, and (2) cantilever beam BC

the internal force $F = 2P/3$ is obtained for the cantilever beam BC



$$\delta_B = \frac{qb^4}{8EI} + \frac{Fb^3}{3EI} = \frac{qb^4}{8EI} + \frac{2Pb^3}{9EI}$$

for the beam AB, θ_A consists of two parts : (1) angle BAB' produced by δ_B, and (2) the bending of beam AB by the load P



$$(\theta_A)_1 = \frac{\delta_B}{a} = \frac{qb^4}{8aEI} + \frac{2Pb^3}{9aEI}$$

the angle due to P is obtained from Case 5 of table G-2, Appendix G with replacing a by $2a/3$ and b by $a/3$

$$(\theta_A)_2 = \frac{P(2a/3)(a/3)(a + a/3)}{6aEI} = \frac{4Pa^2}{81EI}$$

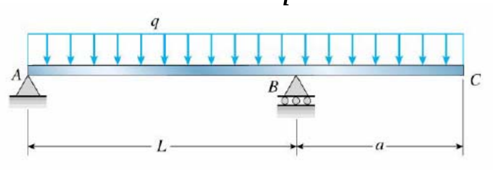
note that in this problem, δ_B is continuous and θ_B does not continuous, i.e. $(\theta_B)_L \neq (\theta_B)_R$

Example 9-9

an overhanging beam ABC supports a uniform load q as shown

determine δ_C

EI = constant

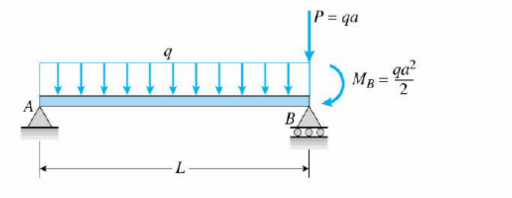


(a)

δ_C may be obtained due to two parts

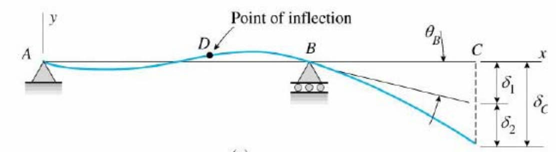
(1) rotation of angle θ_B

(2) a cantilever beam subjected to uniform load q



(b)

firstly, we want to find θ_B



(c)

$$\theta_B = -\frac{qL^3}{24EI} + \frac{M_B L}{3EI}$$

$$= -\frac{qL^3}{24EI} + \frac{qa^2 L}{6EI} = \frac{qL(4a^2 - L^2)}{24EI}$$

then $\delta_1 = a\,\theta_B = \dfrac{qaL(4a^2 - L^2)}{24EI}$

bending of the overhang BC produces an additional deflection δ_2

$$\delta_2 = \frac{qa^4}{8EI}$$

therefore, the total downward deflection of C is

$$\delta_C = \delta_1 + \delta_1 = \frac{qaL(4a^2 - L^2)}{24EI} + \frac{qa^4}{8EI}$$

$$= \frac{qa}{24EI}(a + L)(3a^2 + aL - L^2)$$

for a large, δ_C is downward; for a small, δ_C is upward

for $\delta_C = 0$ $\qquad 3a^2 + aL - L^2 = 0$

$$a = \frac{L(\sqrt{13} - 1)}{6} = 0.4343\,L$$

$a > 0.4343\,L$, δ_C is downward; $a < 0.4343\,L$, δ_C is upward

point D is the point of inflection, the curvature is zero because the bending moment is zero at this point

at point D, $\qquad d^2y/dx^2 = 0$

9.6 Moment-Area Method

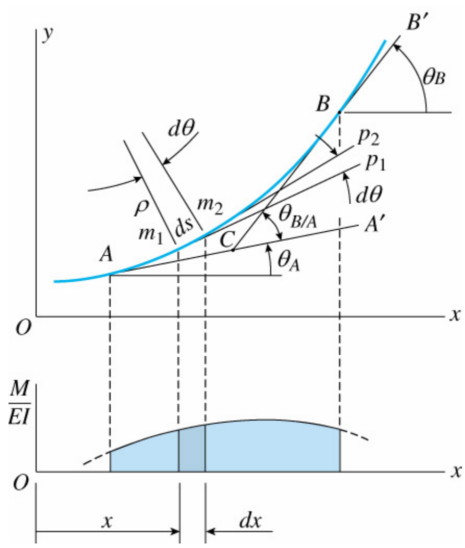
the method is especially suitable when the deflection or angle of rotation at only one point of the beam is desired

consider a segment AB of the beam

denote $\theta_{B/A}$ the difference between θ_B and θ_A

$$\theta_{B/A} = \theta_B - \theta_A$$

consider points m_1 and m_2



$$d\theta = \frac{ds}{\rho} = \frac{Mdx}{EI}$$

Mdx / EI is the area of the shaded strip of the Mdx / EI diagram

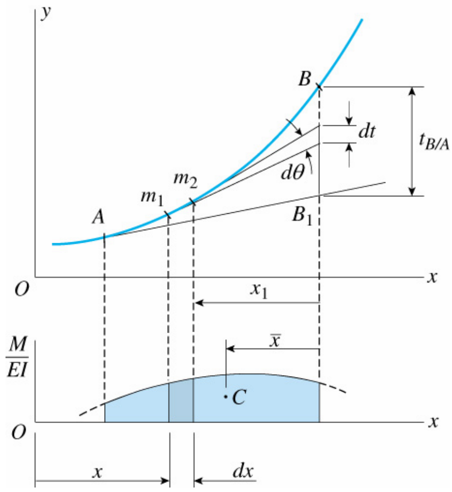
integrating both side between A and B

$$\int_A^B d\theta = \int_A^B \frac{M}{EI} dx$$

$\theta_{B/A} = \theta_B - \theta_A =$ area of the M/EI diagram between A and B

this is the **First moment-area theorem**

next, let us consider the vertical offset $t_{B/A}$ between points B and B_1 (on the tangent of A)



$$\because \quad dt = x_1 \, d\theta = x_1 \frac{Mdx}{EI}$$

integrating between A and B

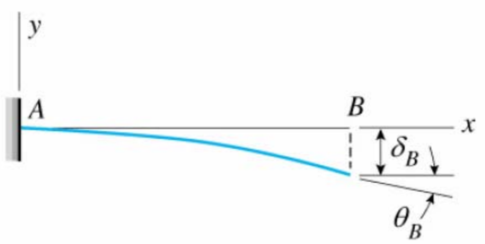
$$\int_A^B dt = \int_A^B x_1 \frac{Mdx}{EI}$$

i.e. $t_{B/A}$ = 1st moment of the area of the M/EI diagram between A and B, evaluated w. r. t. B

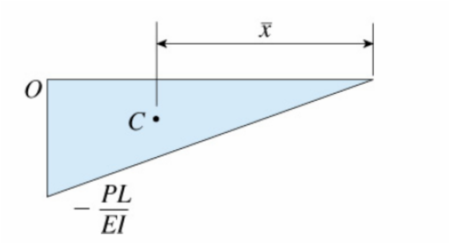
this if the **Second moment-area theorem**

Example 9-10

determine θ_B and δ_B of a cantilever beam AB supporting a



concentrated load P at B



sketch the M/EI diagram first

$$A_1 = -\frac{1}{2} L \frac{PL}{EI} = -\frac{PL^2}{2EI}$$

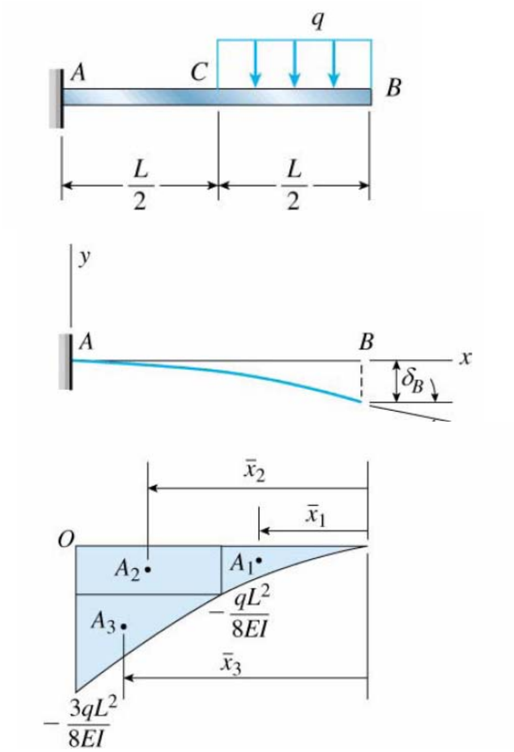
$$\theta_{B/A} = \theta_B - \theta_A = \theta_B = -\frac{PL^2}{2EI} \quad (\quad)$$

$$Q_1 = A_1 x = A_1 \frac{2L}{3} = -\frac{PL^3}{6EI}$$

$$\delta_B = -Q_1 = \frac{PL^3}{6EI} \quad (\downarrow)$$

Example 9-11

determine θ_B and δ_B of a cantilever beam AB supporting a uniform load of intensity q acting over the right-half



sketch the M/EI diagram first

$$A_1 = \frac{1}{3} \frac{L}{2} \left(\frac{qL^2}{8EI}\right) = \frac{qL^3}{48EI}$$

$$A_2 = \frac{L}{2} \left(\frac{qL^2}{8EI}\right) = \frac{qL^3}{16EI}$$

$$A_3 = \frac{1}{2} \frac{L}{2} \left(\frac{qL^2}{4EI}\right) = \frac{qL^3}{16EI}$$

$$\theta_{B/A} = \theta_B = A_1 + A_2 + A_3 = \frac{7qL^3}{16EI} \quad (\circlearrowright)$$

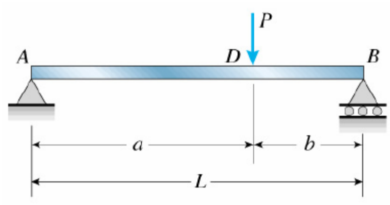
$$\delta_B = t_{B/A}$$

$$= A_1 x_1 + A_2 x_2 + A_3 x_3$$

$$= \frac{qL^3}{EI}\left(\frac{1}{48}\frac{3L}{8} + \frac{1}{16}\frac{3L}{4} + \frac{1}{16}\frac{5L}{6}\right) = \frac{41qL^4}{384EI} \quad (\downarrow)$$

Example 9-12

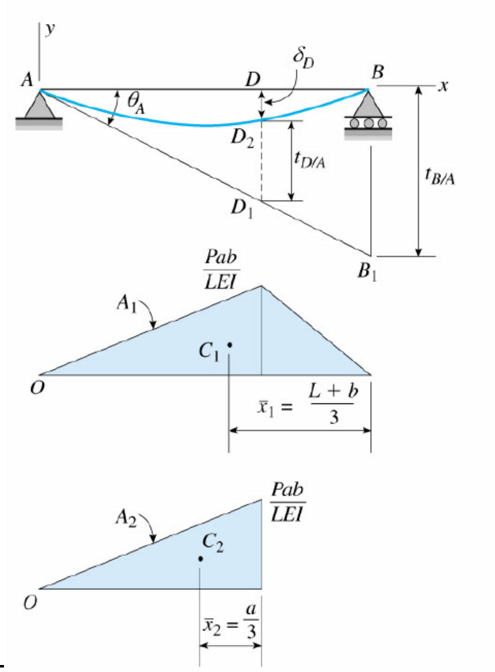
a simple beam ADB supports a concentrated load P as shown determine θ_A and δ_D



$$A_1 = \frac{L}{2}\left(\frac{Pab}{LEI}\right) = \frac{Pab}{2EI}$$

$$t_{B/A} = A_1 \frac{L + b}{3} = \frac{Pab\,(L + b)}{6EI}$$

$$\theta_A = \frac{BB_1}{L} = \frac{Pab\,(L + b)}{6EIL}$$



to find the deflection δ_D at D

$$\delta_D = DD_1 - D_2D_1$$

$$DD_1 = a\,\theta_A = \frac{Pa^2b\,(L + b)}{6EIL}$$

$$D_2D_1 = t_{D/A} = A_2\,x_2$$

$$= \frac{A}{2}\frac{Pab}{EIL}\frac{a}{3} = \frac{Pa^3b}{6EIL}$$

$$\delta_D = \frac{Pa^2b^2}{3EIL}$$

to find the maximum deflection δ_{max} at E, we set $\theta_E = 0$

$$A_3 = \frac{x_1}{2}\frac{Pbx_1}{EIL} = \frac{Pbx_1^{\ 2}}{2EIL}$$

$$\theta_{E/A} = \theta_E - \theta_A = -A_3 = -Pbx_1^{\ 2} / 2EIL$$

$$\theta_A = \frac{Pab\ (L + b)}{6EIL} = \frac{Pbx_1^{\ 2}}{2EIL}$$

then $x_1 = [(L^2 - b^2) / 3]^{½}$

and $\delta_{max} = x_1\ \theta_A - A_3\dfrac{x_1}{3} = \dfrac{Pb}{9\sqrt{3}\,EIL}(L^2 - b^2)^{3/2}$

or δ_{max} = offset of point A from tangent at E

$$\delta_{max} = A_3\frac{2\,x_1}{3} = \frac{Pb}{9\sqrt{3}\,EIL}(L^2 - b^2)^{3/2}$$

Conjugate Beam Method

$$EIv'' = EI\ d\theta/dx = M$$

Integrating

$$\theta = \int \frac{M}{EI}\,dx$$

$$v = \int\int \frac{M}{EI}\,dx\,dx$$

beam theory

$$dM / dx = V \qquad dV / dx = -q$$

$$V = -\int q\,dx \qquad M = -\int\int q\,dx\,dx$$

suppose we have a beam, called conjugate beam, whose length equal to the actual beam, let this beam subjected to so-called "elastic load" of intensity M/EI, then the shear force and bending moment over a portion of the conjugate beam, denoted by $\underline{V}$ and $\underline{M}$, can be obtained

$$\underline{V} = -\int \frac{M}{EI}\,dx \qquad \underline{M} = -\int\int \frac{M}{EI}\,dx\,dx$$

then

(1) the slope at the given section of the beam equals the minus shear force in the corresponding section of the conjugate beam

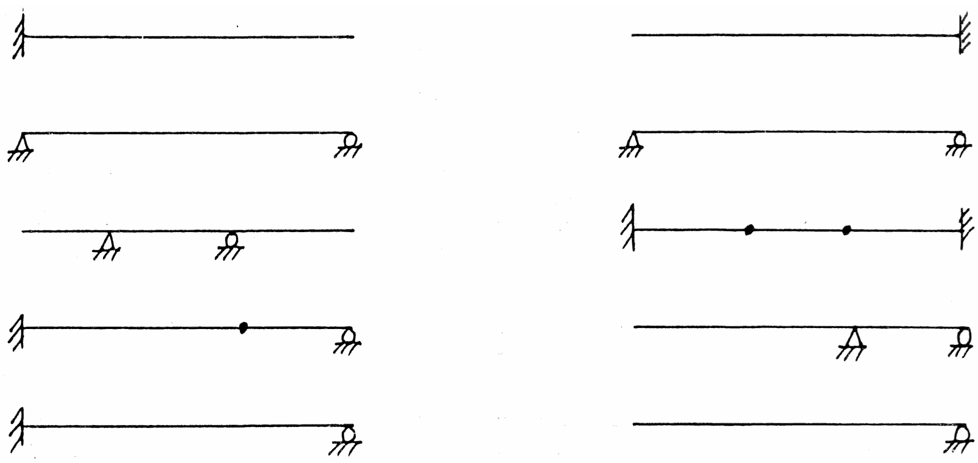
(2) the deflection at the given section of the beam equals the minus bending moment in the corresponding section of the conjugate beam

i.e. $\theta = -\underline{V}$

$\delta = -\underline{M}$

the support conditions between the actual beam and conjugate beam can be found

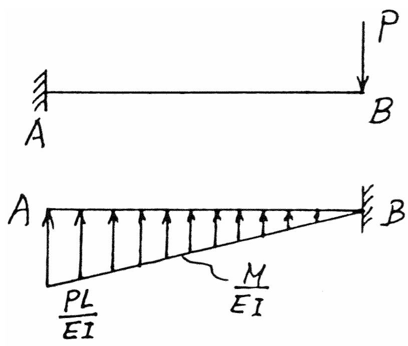
Actual Beam		Conjugate Beam	
fixed end	$\theta = 0$, $v = 0$	$\underline{V} = 0$, $\underline{M} = 0$	**free end**
free end	$\theta \neq 0$, $v \neq 0$	$\underline{V} \neq 0$, $\underline{M} \neq 0$	**fixed end**
simple end	$\theta \neq 0$, $v = 0$	$\underline{V} \neq 0$, $\underline{M} = 0$	**simple end**
interior support	$\theta \neq 0$, $v = 0$	$\underline{V} \neq 0$, $\underline{M} = 0$	**interior hinge**
interior hinge	$\theta \neq 0$, $v \neq 0$	$\underline{V} \neq 0$, $\underline{M} \neq 0$	**interior support**



Example 1

$$\theta_B = -\underline{V}_B = -\frac{1}{2}\frac{PL}{EI}L = -\frac{PL^2}{2EI} \quad (\circlearrowright)$$

$$\delta_B = -\underline{M}_B = -\frac{PL^2}{2EI}\frac{2L}{3} = -\frac{PL^3}{3EI} \quad (\downarrow)$$



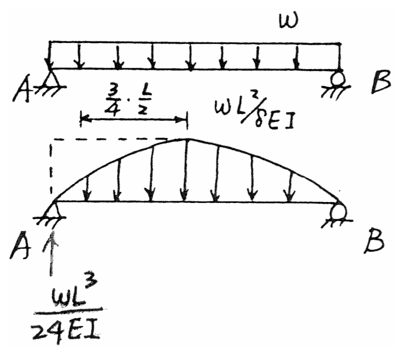
Example 2

$$\theta_A = -\underline{V}_A = -\frac{1}{2}\frac{2L}{3}\frac{wL^2}{8EI} = -\frac{wL^3}{24EI} \quad (\circlearrowright)$$

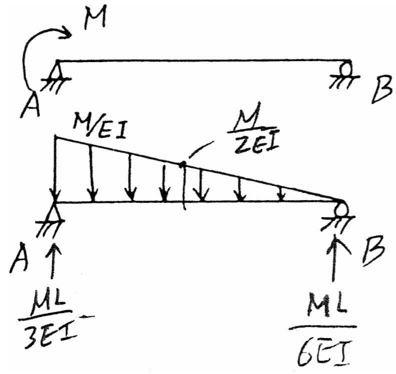
$$\delta_C = -\underline{M}_C = -\frac{wL^3}{24EI}\frac{L}{2} - \frac{wL^2}{8EI}\frac{L}{2}\frac{L}{4}$$

$$+\frac{1}{3}\frac{wL^2}{8EI}\frac{L}{2}\frac{3L}{8} = -\left(\frac{1}{48} - \frac{1}{64} + \frac{1}{128}\right)\frac{wL^4}{EI}$$

$$= \frac{5\,wL^4}{384EI} \quad (\downarrow)$$



Example 3



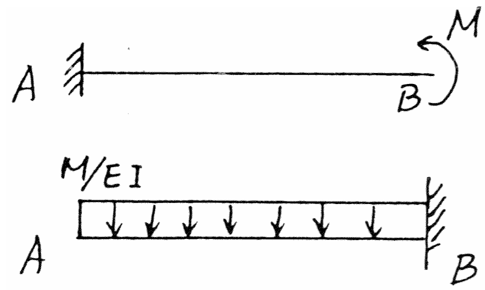
$$\theta_A = -\underline{V}_A = -\underline{R}_A = -\frac{1}{2}\frac{M}{EI}\frac{2L}{3} = -\frac{ML}{3EI} \quad (\circlearrowright)$$

$$\theta_B = -\underline{V}_B = R_B = \frac{1}{2}\frac{M}{EI}\frac{L}{3} = \frac{ML}{6EI} \quad (\quad)$$

$$\delta_C = -\underline{M}_C = -\left(\frac{ML}{6EI}\frac{L}{2} - \frac{1}{2}\frac{L}{2}\frac{M}{2EI}\frac{L}{6}\right)$$

$$= -\left(\frac{1}{12} - \frac{1}{48}\right)\frac{ML^2}{EI} = -\frac{ML^2}{16EI} \quad (\downarrow)$$

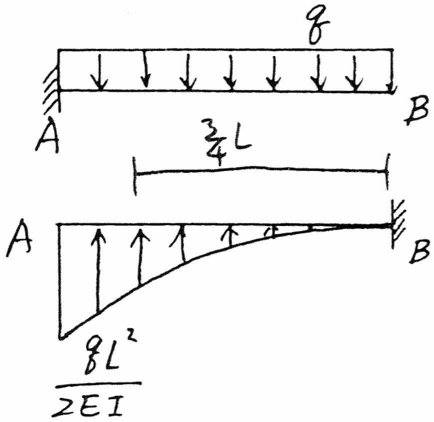
Example 4



$$\theta_B = -\underline{V}_B = \frac{ML}{EI} \quad (\quad)$$

$$\delta_B = -\underline{M}_B = \frac{ML}{EI}\frac{L}{2} = \frac{ML^2}{2EI} \quad (\uparrow)$$

Example 5



$$\theta_B = -\underline{V}_B = -\frac{L}{3}\frac{qL^2}{2EI} = -\frac{qL^3}{6EI} \quad (\circlearrowright)$$

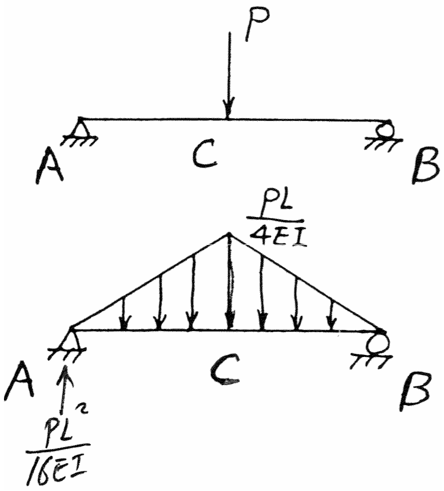
$$\delta_B = -\underline{M}_B = -\frac{qL^3}{6EI}\frac{3L}{4} = -\frac{qL^4}{8EI} \quad (\downarrow)$$

Example 6

$$\theta_A = -\underline{V_A} = -\frac{1}{2}\frac{L}{2}\frac{PL}{4EI} = -\frac{PL^2}{16EI} \qquad (\circlearrowright)$$

$$\delta_C = -\underline{M_C} = -\left(\frac{PL^2}{16EI}\frac{L}{2} - \frac{1}{2}\frac{L}{2}\frac{PL}{4EI}\frac{L}{6}\right)$$

$$= -\frac{PL^3}{EI}\left(\frac{1}{32} - \frac{1}{96}\right) = -\frac{PL^3}{48EI} \qquad (\downarrow)$$

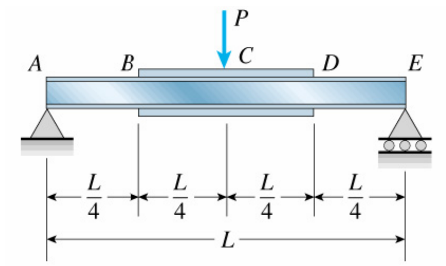


9.7 Nonprismatic Beam

$EI \neq$ constant

Example 9-13

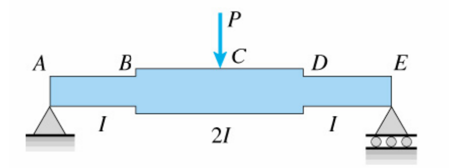
a beam *ABCDE* is supported a concentrated load *P* at midspan as shown



(a)

$I_{BD} = 2\,I_{AB} = 2\,I_{DE} = 2\,I$

determine the deflection curve, θ_A and δ_C

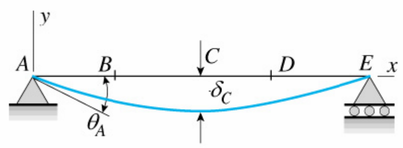


(b)

$$M = \frac{Px}{2} \qquad \left(0 \leqq x \leqq \frac{L}{2}\right)$$

then $\quad EIv'' = Px / 2 \qquad (0 \leqq x \leqq L/4)$

$\qquad E(2I)v'' = Px / 2 \qquad (L/4 \leqq x \leqq L/2)$



(c)

thus $$v' = \frac{Px^2}{4EI} + C_1 \quad (0 \leqq x \leqq L/4)$$

$$v' = \frac{Px^2}{8EI} + C_2 \quad (L/4 \leqq x \leqq L/2)$$

$\because$ $v' = 0$ at $x = L/2$ (symmetric)

$$\therefore \quad C_2 = -\frac{PL^2}{32EI}$$

continuity condition $v'(L/4)^- = v'(L/4)^+$

$$C_1 = -\frac{5PL^2}{128EI}$$

therefore $$v' = -\frac{P}{128EI}(5L^2 - 32x^2) \quad (0 \leqq x \leqq L/4)$$

$$v' = -\frac{P}{32EI}(L^2 - 4x^2) \quad (L/4 \leqq x \leqq L/2)$$

the angle of rotation θ_A is

$$\theta_A = v'(0) = -\frac{5PL^2}{128EI} \quad (↻)$$

integrating the slope equation and obtained

$$v = -\frac{P}{128EI}(5L^2x - \frac{32x^3}{3}) + C_3 \quad (0 \leqq x \leqq L/4)$$

$$v = -\frac{P}{32EI}(L^2x - \frac{4x^3}{3}) + C_4 \quad (L/4 \leqq x \leqq L/2)$$

boundary condition $v(0) = 0$

we get $C_3 = 0$

continuity condition $v(L/4)^- = v(L/4)^+$

we get $C_4 = -\dfrac{PL^3}{768EI}$

therefore the deflection equations are

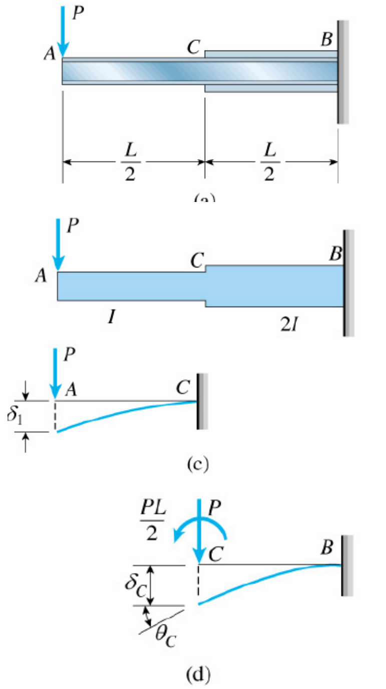
$$v = -\frac{Px}{384EI}(15L^2 - 32x^2) \qquad (0 \leqq x \leqq L/4)$$

$$v = -\frac{P}{768EI}(L^3 + 24L^2x - 32x^3) \qquad (L/4 \leqq x \leqq L/2)$$

the midpoint deflection is obtained

$$\delta_C = -v(L/2) = \frac{3PL^3}{256EI} \qquad (\downarrow)$$

moment-area method and conjugate beam methods can also be used



Example 9-14

a cantilever beam ABC supports a concentrated load P at the free end

$I_{BC} = 2\,I_{AB} = 2\,I$

determine δ_A

denote δ_1 the deflection of A due to C fixed

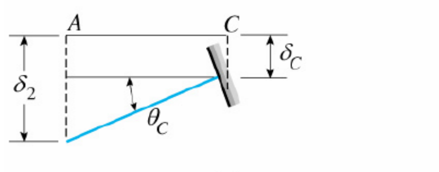
$$\delta_1 = \frac{P(L/2)^3}{3EI} = \frac{PL^3}{24EI}$$

$$\text{and} \quad \delta_C = \frac{P(L/2)^3}{3E(2I)} + \frac{(PL/2)(L/2)^2}{2E(2I)} = \frac{5PL^3}{96EI}$$

$$\theta_C = \frac{P(L/2)^2}{2E(2I)} + \frac{(PL/2)(L/2)}{E(2I)} = \frac{PL^2}{16EI}$$

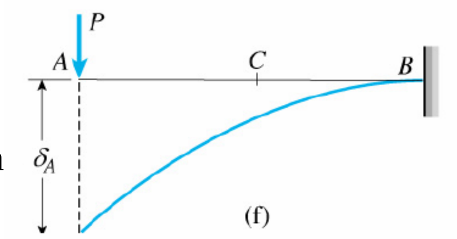
addition deflection at A due to δ_C and θ_C

$$\delta_2 = \delta_C + \theta_C \frac{L}{2} = \frac{5PL^3}{48EI}$$



(e)

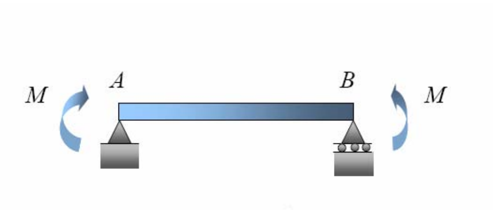
$$\delta_A = \delta_1 + \delta_2 = \frac{5PL^3}{16EI}$$



(f)

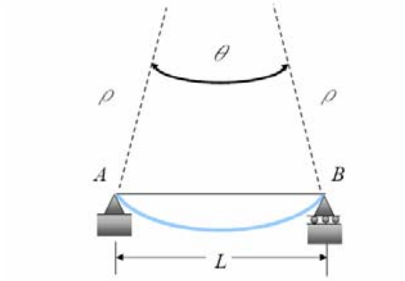
moment-area method and conjugate beam methods can also be used

9.8 Strain Energy of Bending



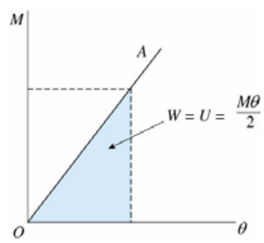
consider a simple beam AB subjected to pure bending under the action of two couples M

the angle θ is



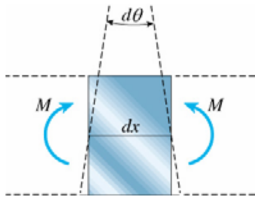
$$\theta = \frac{L}{\rho} = \kappa L = \frac{ML}{EI}$$

if the material is linear elastic, M and θ has linear relation, then



$$W = U = \frac{M\theta}{2} = \frac{M^2L}{2EI} = \frac{EI\theta^2}{2L}$$

for an element of the beam



$$d\theta = \kappa dx = \frac{d^2y}{dx^2} dx$$

$$dU = dW = \frac{M d\theta}{2} = \frac{M^2 dx}{2EI} = \frac{EI(d\theta)^2}{2dx}$$

by integrating throughout the length of the beam

$$U = \int_0^L \frac{M^2 dx}{2EI} = \int_0^L \frac{EI}{2} (\frac{d^2y}{dx^2})^2 dx$$

shear force in beam may produce energy, but for the beam with $L/d > 8$, the strain energy due to shear is relatively small and may be disregarded

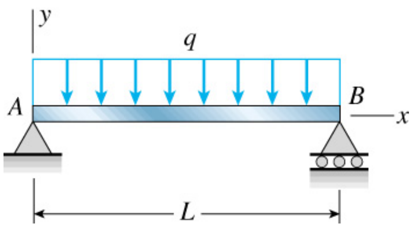
deflection caused by a single load

$$U = W = \frac{P\delta}{2} \qquad U = W = \frac{M_0\theta}{2}$$

$$\delta = \frac{2U}{P} \quad \text{or} \quad \theta = \frac{2U}{M_0}$$

Example 9-15

a simple beam AB of length L supports a uniform load of intensity q



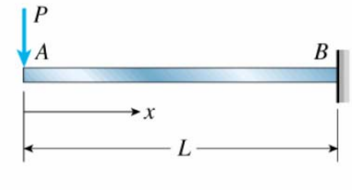
evaluate the strain energy

$$M = \frac{qLx}{2} - \frac{qx^2}{2} = \frac{q}{2}(Lx - x^2)$$

$$U = \int_0^L \frac{M^2dx}{2EI} = \frac{1}{2EI}\int_0^L \left[\frac{q}{2}(Lx - x^2)\right]^2 dx$$

$$= \frac{q^2}{8EI}\int_0^L (L^2x^2 - 2Lx^3 + x^4)dx = q^2L^5 / 240EI$$

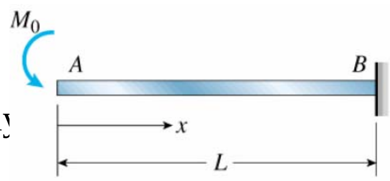
Example 9-16



(a)

a cantilever beam AB is subjected to three different loading conditions

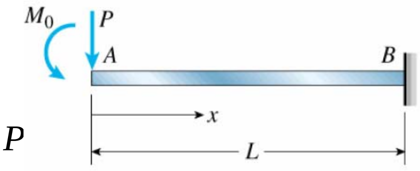
(a) a concentrated load P at its free end



(b)

(b) a moment M_0 at its free end

(c) both P and M_0 acting simultaneousl



(c)

determine δ_A due to loading (a)

determine θ_A due to loading (b)

(a) $M = -Px$

$$U = \int_0^L \frac{M^2dx}{2EI} = \int_0^L \frac{(-Px)^2dx}{2EI} = \frac{P}{6.}$$

$$W = U \qquad \frac{P\delta_A}{2} = \frac{P^2L^3}{6EI} \qquad \delta_A = \frac{PL^3}{3EI}$$

(b) $M = -M_0$

$$U = \int_0^L \frac{M^2dx}{2EI} = \int_0^L \frac{(-M_0)^2dx}{2EI} = \frac{M_0^2L}{2EI}$$

$$W = U \qquad \frac{M_0\theta_A}{2} = \frac{M_0^2L}{2EI} \qquad \theta_A = \frac{M_0L}{EI}$$

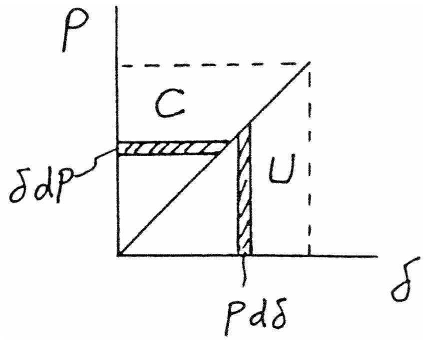
(c) $M = -Px - M_0$

$$U = \int_0^L \frac{M^2dx}{2EI} = \int_0^L \frac{(-Px - M_0)^2dx}{2EI} = \frac{P^2L^3}{6EI} + \frac{PM_0L^2}{2EI} + \frac{M_0^2L}{2EI}$$

$$W = U \qquad \frac{P\delta_A}{2} + \frac{M_0\theta_A}{2} = \frac{P^2L^3}{6EI} + \frac{PM_0L^2}{2EI} + \frac{M_0^2L}{2EI}$$

1 equation for two unknowns δ_A and θ_A

9.9 Castigliano's Theorem (Energy Method)



$$dU = P\,d\delta \qquad \frac{dU}{d\delta} = P$$

$$dC = \delta\,dP \qquad \frac{dC}{dP} = \delta$$

where C is complementary strain energy

for linear elastic materials $C = U$

then we have $\dfrac{dU}{dP} = \delta$

similarly $\dfrac{dU}{dM} = \theta$

for both P and M acting simultaneously, $U = U(P, M)$

$$\frac{\partial U}{\partial P} = \delta \qquad \frac{\partial U}{\partial M} = \theta$$

in example 9-16 (c)

$$U = \frac{P^2L^3}{6EI} + \frac{PM_0L^2}{2EI} + \frac{M_0{}^2L}{2EI}$$

$$\delta = \frac{\partial U}{\partial P} = \frac{PL^3}{6EI} + \frac{M_0L^2}{2EI}$$

$$\theta = \frac{\partial U}{\partial M} = \frac{PL^2}{2EI} + \frac{M_0L}{EI}$$

in general relationship

$$\delta_i = \frac{\partial U}{\partial P_i} \qquad \text{Castigliano's Theorem}$$

$$\delta_i = \frac{\partial U}{\partial P_i} = \frac{\partial}{\partial P_i} \int \frac{M^2dx}{2EI} = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial P_i}\right)dx$$

this is the modified Castigliano's Theorem

in example 9-16 (c)

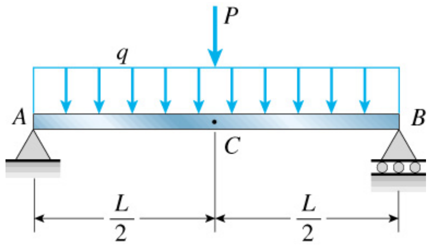
$$M = -Px - M_0$$

$$\frac{\partial M}{\partial P} = -x \qquad \frac{\partial U}{\partial M_0} = -1$$

$$\delta = \frac{1}{EI} \int (-Px - M_0)(-x)dx = \frac{PL^3}{6EI} + \frac{M_0L^2}{2EI}$$

$$\theta = \frac{1}{EI} \int (-Px - M_0)(-1)dx = \frac{PL^2}{2EI} + \frac{M_0L}{EI}$$

Example 9-17

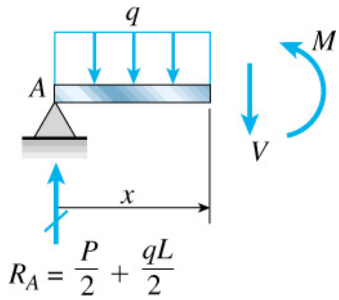


a simple beam *AB* supports a uniform load $q = 20$ kN/m, and a concentrated load $P = 25$ kN

$L = 2.5$ m $E = 210$ GPa

$I = 31.2 \times 10^2$ cm^4

determine δ_C



$$M = \frac{Px}{2} + \frac{qLx}{2} - \frac{qx^2}{2}$$

method (1)

$$U = \int \frac{M^2 dx}{2EI} = 2 \int_0^{L/2} \frac{1}{2EI}\left(\frac{Px}{2} + \frac{qLx}{2} - \frac{qx^2}{2}\right)^2 dx$$

$$= \frac{P^2L^3}{96EI} + \frac{5PqL^4}{384EI} + \frac{q^2L^5}{240EI}$$

$$\delta_C = \frac{\partial U}{\partial P} = \frac{PL^3}{48EI} + \frac{5qL^4}{384EI}$$

method (2)

$$\partial M / \partial P = x/2$$

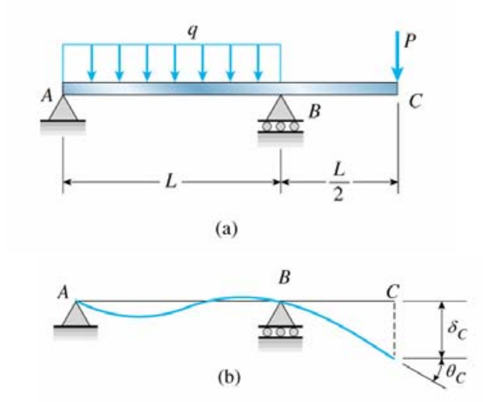
$$\delta_C = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial P}\right) dx = 2 \int_0^{L/2} \frac{1}{EI}\left(\frac{Px}{2} + \frac{qLx}{2} - \frac{qx^2}{2}\right)\left(\frac{x}{2}\right) dx$$

$$= \frac{PL^3}{48EI} + \frac{5qL^4}{384EI}$$

$$= 1.24 \text{ mm} + 1.55 \text{ mm} = 2.79 \text{ mm}$$

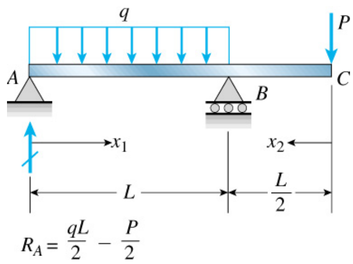
Example 9-18

a overhanging beam *ABC* supports a uniform load and a concentrated load as shown



determine δ_C and θ_C

the reaction at *A* due to the loading is



$$R_A = \frac{qL}{2} - \frac{P}{2}$$

$$M_{AB} = R_A x_1 - \frac{qx_1^2}{2}$$

$$= \frac{qLx_1}{2} - \frac{Px_1}{2} - \frac{qx_1^2}{2} \quad (0 \leqq x_1 \leqq L)$$

$$M_{BC} = -Px_2 \quad (0 \leqq x_1 \leqq L/2)$$

then the partial derivatives are

$$\partial M_{AB} / \partial P = -x_1/2 \qquad (0 \leqq x_1 \leqq L)$$

$$\partial M_{BC} / \partial P = -x_2 \qquad (0 \leqq x_2 \leqq L/2)$$

$$\delta_C = \int (M/EI)(\partial M / \partial P)dx$$

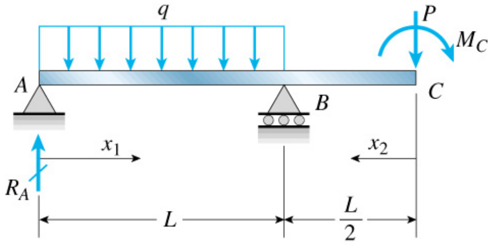
$$= \int_0^L (M_{AB}/EI)(\partial M_{AB}/\partial P)dx + \int_0^{L/2} (M_{BC}/EI)(\partial M_{BC}/\partial P)dx$$

$$= \frac{1}{EI}\int_0^L \left(\frac{qLx_1}{2} - \frac{Px_1}{2} - \frac{qx_1^2}{2}\right)\left(-\frac{x_1}{2}\right)dx_1 + \frac{1}{EI}\int_0^{L/2} (-Px_2)(-x_2)dx_2$$

$$= \frac{PL^3}{8EI} - \frac{qL^4}{48EI}$$

to determine the angle θ_C, we place a couple of moment M_C at C

$$R_A = \frac{qL}{2} - \frac{P}{2} - \frac{M_C}{L}$$



$$M_{AB} = R_A x_1 - \frac{qx_1^2}{2}$$

$$= \frac{qLx_1}{2} - \frac{Px_1}{2} - \frac{M_C x_1}{L} - \frac{qx_1^2}{2} \quad (0 \leqq x_1 \leqq L)$$

$$M_{BC} = -Px_2 - M_C \quad (0 \leqq x_1 \leqq L/2)$$

then the partial derivatives are

$$\partial M_{AB} / \partial M_C = -x_1/L \quad (0 \leqq x_1 \leqq L)$$

$$\partial M_{BC} / \partial M_C = -1 \quad (0 \leqq x_2 \leqq L/2)$$

$$\theta_C = \int (M/EI)(M/M_C)dx$$

$$= \int_0^L (M_{AB}/EI)(M_{AB}/M_C)dx + \int_0^{L/2} (M_{BC}/EI)(M_{BC}/M_C)dx$$

$$= \frac{1}{EI}\int_0^L \left(\frac{qLx_1}{2} - \frac{Px_1}{2} - \frac{M_C x_1}{L} - \frac{qx_1^2}{2}\right)\left(-\frac{x_1}{L}\right)dx_1$$

$$+ \frac{1}{EI}\int_0^{L/2} (-Px_2 - M_C)(-1)dx_2$$

since M_C is a virtual load, set $M_C = 0$, after integrating θ_C is obtained

$$\theta_C = \frac{7PL^2}{24EI} - \frac{qL^4}{24EI}$$

9.10 Deflections Produced by Impact

9.11 Temperature Effects