MECHANICS OF SOLIDS - BEAMS
TUTORIAL 3

THE DEFLECTION OF BEAMS

This is the third tutorial on the bending of beams. You should judge your progress by completing the self assessment exercises.

On completion of this tutorial you should be able to solve the slope and deflection of the following types of beams.

- A cantilever beam with a point load at the end.
- A cantilever beam with a uniformly distributed load.
- A simply supported beam with a point load at the middle.
- A simply supported beam with a uniformly distributed load.

You will also learn and apply Macaulay's method to the solution for beams with a combination of loads.

Those who require more advanced studies may also apply Macaulay's method to the solution of ENCASTRÉ.

It is assumed that students doing this tutorial already know how to find the bending moment in various types of beams. This information is contained in tutorial 2.

DEFLECTION OF BEAMS

1. GENERAL THEORY

When a beam bends it takes up various shapes such as that illustrated in figure 1. The shape may be superimposed on an x – y graph with the origin at the left end of the beam (before it is loaded). At any distance x metres from the left end, the beam will have a deflection y and a gradient or slope dy/dx and it is these that we are concerned with in this tutorial.

We have already examined the equation relating bending moment and radius of curvature in a beam, namely $\frac{M}{I} = \frac{E}{R}$

M is the bending moment.
I is the second moment of area about the centroid.
E is the modulus of elasticity and
R is the radius of curvature.

Rearranging we have $\frac{1}{R} = \frac{M}{EI}$

Figure 1 illustrates the radius of curvature which is defined as the radius of a circle that has a tangent the same as the point on the x-y graph.

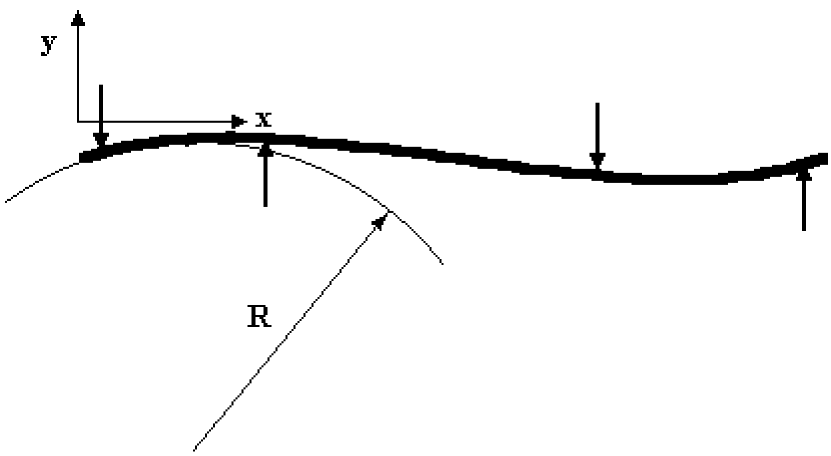

Figure 1

Mathematically it can be shown that any curve plotted on x - y graph has a radius of curvature of defined as

$$\frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\left[1 + \frac{dy}{dx}\right]^{\frac{3}{2}}}$$

In beams, R is very large and the equation may be simplified without loss of accuracy to

$$\frac{1}{R}=\frac{d^2x}{dy^2}$$

hence
$$\frac{d^2x}{dy^2}=\frac{M}{EI}$$

or
$$M = EI\frac{d^2x}{dy^2}\text{............(1A)}$$

The product EI is called the flexural stiffness of the beam.

In order to solve the slope (dy/dx) or the deflection (y) at any point on the beam, an equation for M in terms of position x must be substituted into equation (1A). We will now examine this for the 4 standard cases.

2. CASE 1 - CANTILEVER WITH POINT LOAD AT FREE END.

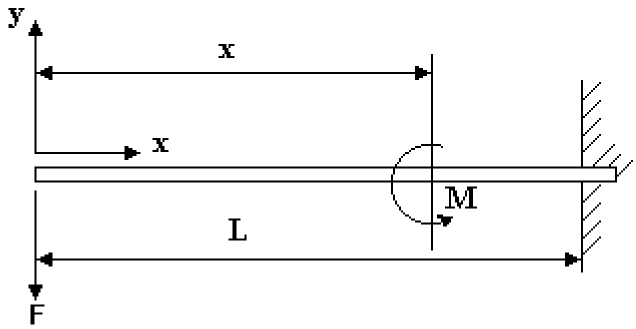

Figure 2

The bending moment at any position x is simply -Fx. Substituting this into equation 1A we have

$$EI\frac{d^2y}{dx^2} = -Fx$$

Integrate wrt x and we get $\quad EI\frac{dy}{dx} = -\frac{Fx^2}{2} + A \ldots\ldots\ldots\ldots\ldots\ldots(2A)$

Integrate again and we get $\quad EIy = -\frac{Fx^3}{6} + Ax + B \ldots\ldots\ldots\ldots\ldots\ldots(2B)$

A and B are constants of integration and must be found from the boundary conditions.

These are at x = L, y = 0 (no deflection)
at x = L, dy/dx = 0 (gradient horizontal)

Substitute x = L and dy/dx = 0 in equation 2A. This gives

$EI(0) = -\frac{FL^2}{2} + A \qquad$ hence $A = \frac{FL^2}{2}$

substitute $A = \frac{FL^2}{2}$, $y = 0$ and $x = L$ nto equation 2B and we get

$EI(0) = -\frac{FL^3}{6} + \frac{FL^3}{2} + B \qquad$ hence $B = -\frac{FL^3}{3}$

substitute $A = \frac{FL^2}{2}$ and $B = -\frac{FL^3}{3}$ into equations 2A and 2B and the complete equations are

$$EI\frac{dy}{dx} = -\frac{Fx^2}{2} + \frac{FL^2}{2} \ldots\ldots\ldots\ldots\ldots\ldots(2C)$$

$$EIy = -\frac{Fx^3}{6} + \frac{FL^2x}{2} - \frac{FL^3}{3} \ldots\ldots\ldots\ldots\ldots\ldots(2D)$$

The main point of interest is the slope and deflection at the free end where x=0. Substituting x= 0 into (2C) and (2D) gives the standard equations.

Slope at free end $\qquad \frac{dy}{dx} = \frac{FL^2}{2EI} \ldots\ldots\ldots\ldots\ldots\ldots(2E)$

Deflection at free end $\qquad y = -\frac{FL^3}{3EI} \ldots\ldots\ldots\ldots\ldots\ldots(2F)$

WORKED EXAMPLE No.1

A cantilever beam is 4 m long and has a point load of 5 kN at the free end. The flexural stiffness is 53.3 MNm^2. Calculate the slope and deflection at the free end.

SOLUTION

i. Slope

Using formula 2E we have $\frac{dy}{dx} = \frac{FL^2}{2EI} = \frac{5000 \times 4^2}{2 \times 53.3 \times 10^6} = 750 \times 10^{-6}$ (no units)

ii. Deflection

Using formula 2F we have $y = -\frac{FL^3}{3EI} = -\frac{5000 \times 4^3}{3 \times 53.3 \times 10^6} = -0.002 \text{ m}$

The deflection is 2 mm downwards.

SELF ASSESSMENT EXERCISE No.1

1. A cantilever beam is 6 m long and has a point load of 20 kN at the free end. The flexural stiffness is 110 MNm^2. Calculate the slope and deflection at the free end. (Answers 0.00327 and -13 mm).

2. A cantilever beam is 5 m long and has a point load of 50 kN at the free end. The deflection at the free end is 3 mm downwards. The modulus of elasticity is 205 GPa. The beam has a solid rectangular section with a depth 3 times the width. (D= 3B). Determine

   i. the flexural stiffness. (694.4 MNm^2)

   ii. the dimensions of the section. (197 mm wide and 591 mm deep).

3. CASE 2 - CANTILEVER WITH A UNIFORMLY DISTRIBUTED LOAD.

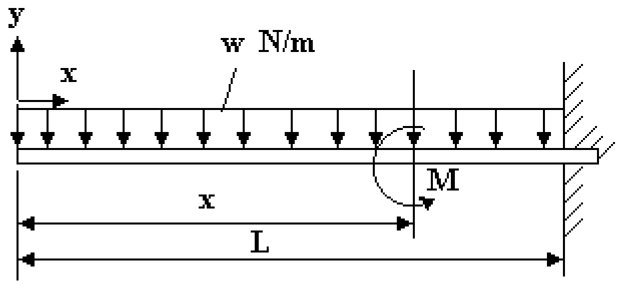

Figure 3

The bending moment at position x is given by M = -wx2/2. Substituting this into equation 1A we have

$$EI\frac{d^2y}{dx^2} = -w\frac{x^2}{2}$$

Integrate wrt x and we get $EI\frac{dy}{dx} = -\frac{wx^3}{6} + A$ ....(3A)

Integrate again and we get $EIy = -\frac{wx^4}{24} + Ax + B$ ....(3B)

A and B are constants of integration and must be found from the boundary conditions. These are

at x = L , y = 0 (no deflection)
at x = L, dy/dx = 0 (horizontal)

Substitute x = L and dy/dx = 0 in equation 3A and we get

$$EI(0) = -\frac{wL^3}{6} + A \quad \text{hence} \quad A = \frac{wL^3}{6}$$

Substitute this into equation 3B with the known solution y = 0 and x = L results in

$$EI(0) = -\frac{wL^4}{24} + \frac{wL^4}{6} + B \quad \text{hence } B = -\frac{wL^4}{8}$$

Putting the results for A and B into equations 3A and 3B yields the complete equations

$$EI\frac{dy}{dx} = -\frac{wx^3}{6} + \frac{wL^3}{6} \text{ ....(3C)}$$

$$EIy = -\frac{wx^4}{24} + \frac{wL^3x}{6} - \frac{wL^4}{8} \text{ ....(3D)}$$

The main point of interest is the slope and deflection at the free end where x=0. Substituting x= 0 into (3C) and (3D) gives the standard equations.

Slope at free end $\frac{dy}{dx} = \frac{wL^3}{6EI}$ ....(3E)

Deflection at free end $y = -\frac{wL^4}{8EI}$ ....(3F)

WORKED EXAMPLE No.2

A cantilever beam is 4 m long and has a u.d.l. of 300 N/m. The flexural stiffness is 60 MNm^2. Calculate the slope and deflection at the free end.

SOLUTION

i. Slope

From equation 3E we have $\frac{dy}{dx} = \frac{wL^3}{6EI} = \frac{300 \times 4^3}{6 \times 60 \times 10^6} = 53.3 \times 10^{-6}$ (no units)

ii. Deflection

From equation 3F we have $y = -\frac{wL^4}{8EI} = -\frac{300 \times 4^4}{8 \times 60 \times 10^6} = -0.00016 \text{ m}$

Deflection is 0.16 mm downwards.

SELF ASSESSMENT EXERCISE No.2

1. A cantilever is 6 m long with a u.d.l. of 1 kN/m. The flexural stiffness is 100 MNm^2. Calculate the slope and deflection at the free end.
(360 x 10^{-6} and -1.62 mm)

2. A cantilever beam is 5 m long and carries a u.d.l. of 8 kN/m. The modulus of elasticity is 205 GPa and beam is a solid circular section. Calculate

i. the flexural stiffness which limits the deflection to 3 mm at the free end.
(208.3 MNm^2).

ii. the diameter of the beam. (379 mm).

4. CASE 3 - SIMPLY SUPPORTED BEAM WITH POINT LOAD IN MIDDLE.

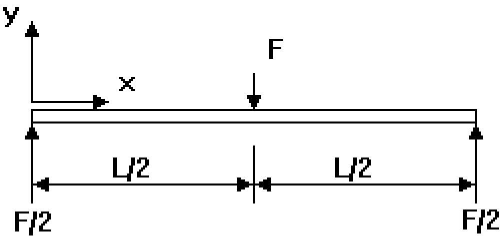

Figure 4

The beam is symmetrical so the reactions are F/2. The bending moment equation will change at the centre position but because the bending will be symmetrical each side of the centre we need only solve for the left hand side.

The bending moment at position x up to the middle is given by M = Fx/2. Substituting this into equation 1A we have

$$EI\frac{d^2y}{dx^2} = \frac{Fx}{2}$$

$$\text{Integrate wrt x once} \qquad EI\frac{dy}{dx} = \frac{Fx^2}{4} + A \ldots\ldots\ldots\ldots\ldots\ldots(4A)$$

$$\text{Integrate wrt x again} \qquad EIy = \frac{Fx^3}{12} + Ax + B \ldots\ldots\ldots\ldots\ldots(4B)$$

A and B are constants of integration and must be found from the boundary conditions. These are

at x = 0 , y = 0 (no deflection at the ends)
at x = L/2, dy/dx = 0 (horizontal at the middle)

putting x = L/2 and dy/dx = 0 in equation 4A results in

$$EI(0) = \frac{FL^2}{16} + A \qquad \text{hence } A = -\frac{FL^2}{16}$$

$$\text{substitute } A = -\frac{FL^2}{16}, \text{ y = 0 and x = 0 nto equation 4B and we get}$$

$$EI(0) = B \qquad \text{hence } B = 0$$

$$\text{substitute } A = -\frac{FL^2}{16} \text{ and } B = 0 \text{ into equations 4A and 4B and the complete equations are}$$

$$EI\frac{dy}{dx} = \frac{Fx^2}{4} - \frac{FL^2}{16} \ldots\ldots\ldots\ldots\ldots\ldots(4C)$$

$$EIy = \frac{Fx^3}{12} - \frac{FL^3x}{16} \ldots\ldots\ldots\ldots\ldots\ldots(4D)$$

The main point of interest is the slope at the ends and the deflection at the middle .
Substituting x = 0 into (4C) gives the standard equation for the slope at the left end. The slope at the right end will be equal but of opposite sign.

$$\text{Slope at ends} \qquad \frac{dy}{dx} = \pm\frac{FL^2}{16EI} \ldots\ldots\ldots(4E)$$

The slope is negative on the left end but will be positive on the right end.
Substituting x = L/2 into equation 4D gives the standard equation for the deflection at the middle:

$$\text{Deflection at middle} \qquad y = -\frac{FL^3}{48EI} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4F)$$

WORKED EXAMPLE No.3

A simply supported beam is 8 m long with a load of 500 kN at the middle. The deflection at the middle is 2 mm downwards. Calculate the gradient at the ends.

SOLUTION

From equation 4F we have

$$y = -\frac{FL^3}{48EI} \quad \text{and y is 2 mm down so } y = -0.002 \text{ m}$$

$$-0.002 = -\frac{500 \text{ x } 8^2}{48EI}$$

$$EI = 2.667 \text{ x } 10^9 \text{ Nm}^2 \quad \text{or} \quad 2.667 \text{ GNm}^2$$

From equation 4E we have

$$\frac{dy}{dx} = \pm\frac{FL^2}{16EI} = \pm\frac{500\,000 \text{ x } 8^2}{16 \text{ x } 2.667 \text{ x } 10^9} = 750 \text{ x } 10^{-6} \quad \text{(no units)}$$

The gradient will be negative at the left end and positive at the right end.

SELF ASSESSMENT EXERCISE No.3

1. A simply supported beam is 4 m long and has a load of 200 kN at the middle. The flexural stiffness is 300 MNm^2. Calculate the slope at the ends and the deflection at the middle.
(0.000667 and -0.89 mm).

2. A simply supported beam is made from a hollow tube 80 mm outer diameter and 40 mm inner diameter. It is simply supported over a span of 6 m. A point load of 900 N is placed at the middle. Find the deflection at the middle if E=200 GPa.
(-10.7 mm).

3. Find the flexural stiffness of a simply supported beam which limits the deflection to 1 mm at the middle. The span is 2 m and the point load is 200 kN at the middle.
(33.3 MNm^2).

5. CASE 4 - SIMPLY SUPPORTED BEAM WITH A UNIFORMLY DISTRIBUTED LOAD.

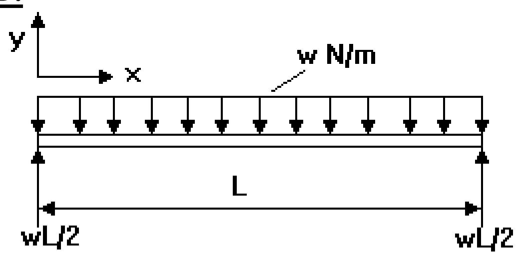

Figure 5

The beam is symmetrical so the reactions are wL/2. The bending moment at position x is

$$M = \frac{wLx}{2} - \frac{wx^2}{2}$$

Substituting this into equation 1A we have

$$EI\frac{d^2y}{dx^2} = \frac{wLx}{2} - \frac{wx^2}{2}$$

$$\text{Integrate wrt x once} \quad EI\frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} + A.................(5A)$$

$$\text{Integrate wrt x again} \quad EIy = \frac{wLx^3}{12} - \frac{wx^4}{24} + Ax + B.............(5B)$$

A and B are constants of integration and must be found from the boundary conditions. These are at x = 0 , y = 0 (no deflection at the ends)
at x = L/2, dy/dx = 0 (horizontal at the middle)

Putting x = L/2 and dy/dx = 0 in equation 5A results in

$$EI(0) = \frac{wL^3}{16} - \frac{wL^3}{48} + A \qquad \text{hence } A = -\frac{wL^3}{24}$$

$$\text{substitute } A = -\frac{wL^3}{24}, \ y = 0 \text{ and } x = 0 \text{ nto equation 5B and we get}$$

$$EI(0) = B \qquad \text{hence } B = 0$$

$$\text{substitute } A = -\frac{wL^3}{24} \text{ and } B = 0 \text{ into equations 5A and 5B and the complete equations are}$$

$$EI\frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} - \frac{wL^2}{24}.................(4C)$$

$$EIy = \frac{wLx^3}{12} - \frac{wx^4}{24} + \frac{wL^3x}{24}.................(4D)$$

The main point of interest is the slope at the ends and the deflection at the middle.

Substituting x = 0 into (5C) gives the standard equation for the slope at the left end. The slope at the right end will be equal but of opposite sign.

$$\text{Slope at free end} \qquad \frac{dy}{dx} = \pm\frac{wL^3}{24EI}.............(5E)$$

The slope is negative on the left end but will be positive on the right end.

Substituting x= L/2 into equation 5D gives the standard equation for the deflection at the middle:

$$\text{Deflection at middle} \qquad y = -\frac{5wL^4}{384EI}...............(5F)$$

WORKED EXAMPLE No.4

A simply supported beam is 8 m long with a u.d.l. of 5000 N/m. Calculate the flexuralstiffness which limits the deflection to 2 mm at the middle. Calculate the gradient at the ends.

SOLUTION

Putting y = -0.002 m into equation 5F we have

$$y = -\frac{5wL^4}{384EI}$$

$$-0.002 = -\frac{5 \text{ x } 5000 \text{ x } 4^4}{384 \text{ x EI}} \qquad EI = 133.3 \text{ x } 10^6 \text{ Nm}^2 \text{ or } 133.3 \text{ MNm}^2$$

From equation 5E we have

$$\frac{dy}{dx} = \pm\frac{5000 \text{ x } 8^3}{24 \text{ x } 133.3 \text{ x } 10^6} = \pm 800 \text{ x } 10^{-6} \text{ (no units)}$$

The gradient will be negative at the left end and positive at the right end.

SELF ASSESSMENT EXERCISE No.4

1. A simply supported beam is 4 m long with a u.d.l. of 200 N/m. The flexural stiffness is 100 MNm^2. Calculate the slope at the ends and the deflection at the middle.
(5.33 x 10^{-6}) and -6.67 x 10^{-6} m).

2. A simply supported beam is made from a hollow tube 80 mm outer diameter and 40 mm inner diameter. It is simply supported over a span of 6 m. The density of the metal is 7300 kg/m^3. E=200 GPa. Calculate the deflection at the middle due to the weight of the beam.
(-12 mm)

3. Find the flexural stiffness of a simply supported beam which limits the deflection to 1 mm at the middle. The span is 2 m and the u.d.l. is 400 N/m.
(83.3 kNm^2)

6. THE THEORY OF SUPERPOSITION FOR COMBINED LOADS.

This theory states that the slope and deflection of a beam at any point is the sum of the slopes and deflections which would be produced by each load acting on its own. For beams with combinations of loads which are standard cases we only need to use the standard formulae. This is best explained with a worked example.

WORKED EXAMPLE No.5

A cantilever beam is 4m long with a flexural stiffness of 20 MNm^2. It has a point load of 1 kN at the free end and a u.d.l. of 300 N/m along its entire length. Calculate the slope and deflection at the free end.

SOLUTION

For the point load only

$$y = -\frac{FL^3}{3EI} = -\frac{1000 \times 4^3}{3 \times 20 \times 10^6} = -0.00106 \text{ m or } -1.06 \text{ mm}$$

For the u.d.l. only

$$y = -\frac{wL^4}{48EI} = -\frac{300 \times 4^4}{8 \times 20 \times 10^6} = -0.00048 \text{ m or } -0.48 \text{ mm}$$

The total deflection is hence **y= - 1.54 mm.**

For the point load only

$$\frac{dy}{dx} = \frac{FL^2}{2EI} = \frac{1000 \times 4^2}{2 \times 20 \times 10^6} = 400 \times 10^{-6}$$

For the u.d.l. only

$$\frac{dy}{dx} = \frac{wL^3}{6EI} = \frac{300 \times 4^3}{6 \times 20 \times 10^6} = 160 \times 10^{-6}$$

The total slope is hence **dy/dx =560 x 10^{-6}.**

7. MACAULAY'S METHOD

When the loads on a beam do not conform to standard cases, the solution for slope and deflection must be found from first principles. Macaulay developed a method for making the integrations simpler.

The basic equation governing the slope and deflection of beams is

$$EI\frac{d^2y}{dx^2} = M \text{ Where M is a function of x.}$$

When a beam has a variety of loads it is difficult to apply this theory because some loads may be within the limits of x during the derivation but not during the solution at a particular point. Macaulay's method makes it possible to do the integration necessary by placing all the terms containing x within a square bracket and **integrating the bracket, not x.** During evaluation, any bracket with a negative value is ignored because a negative value means that the load it refers to is not within the limit of x. The general method of solution is conducted as follows. Refer to figure 6. In a real example, the loads and reactions would have numerical values but for the sake of demonstrating the general method we will use algebraic symbols. This example has only point loads.

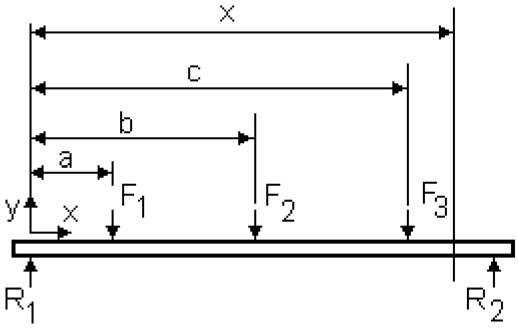

Figure 6

1. Write down the bending moment equation placing x on the extreme right hand end of the beam so that it contains all the loads. write all terms containing x in a square bracket.

$$EI\frac{d^2y}{dx^2} = M = R_1[x] - F_1[x-a] - F_2[x-b] - F_3[x-c]$$

2. Integrate once treating the square bracket as the variable.

$$EI\frac{dy}{dx} = R_1\frac{[x]^2}{2} - F_1\frac{[x-a]^2}{2} - F_2\frac{[x-b]^2}{2} - F_3\frac{[x-c]^2}{2} + A$$

3. Integrate again using the same rules.

$$EIy = R_1\frac{[x]^3}{6} - F_1\frac{[x-a]^3}{6} - F_2\frac{[x-b]^3}{6} - F_3\frac{[x-c]^3}{6} + Ax + B$$

4. Use boundary conditions to solve A and B.

5. Solve slope and deflection by putting in appropriate value of x. IGNORE any brackets containing negative values.

WORKED EXAMPLE No.6

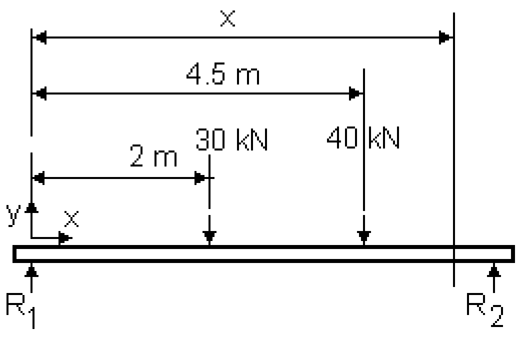

Figure 7

The beam shown is 7 m long with an E I value of 200 MNm^2. Determine the slope and deflection at the middle.

SOLUTION

First solve the reactions by taking moments about the right end.

30 x 5 + 40 x 2.5 = 7 R_1 hence R_1 = 35.71 kN

R_2 = 70 - 35.71 = 34.29 kN

Next write out the bending equation.

$$EI\frac{d^2y}{dx^2} = M = 35710[x] - 30000[x-2] - 40000[x-4.5]$$

Integrate once treating the square bracket as the variable.

$$EI\frac{dy}{dx} = 35710\frac{[x]^2}{2} - 30000\frac{[x-2]^2}{2} - 40000\frac{[x-4.5]^2}{2} + A \text{ .....(1)}$$

Integrate again

$$EIy = 35710\frac{[x]^3}{6} - 30000\frac{[x-2]^3}{6} - 40000\frac{[x-4.5]^3}{6} + Ax + B \text{ ......(2)}$$

BOUNDARY CONDITIONS

x = 0, y = 0 and x = 7 y = 0

Using equation 2 and putting x = 0 and y = 0 we get

$$EI(0) = 35710\frac{[0]^3}{6} - 30000\frac{[0-2]^3}{6} - 40000\frac{[0-4.5]^3}{6} + A(0) + B$$

Ignore any bracket containing a negative value.

$$0 = 0 - 0 - 0 + 0 + B \quad \text{hence } B = 0$$

Using equation 2 again but this time x=7 and y = 0

$$EI(0) = 35710\frac{[7]^3}{6} - 30000\frac{[7-2]^3}{6} - 40000\frac{[7-4.5]^3}{6} + A(7) + 0$$

Evaluate A and A = -187400

Now use equations 1 and 2 with x = 3.5 to find the slope and deflection at the middle.

$$EI\frac{dy}{dx} = 35710\frac{[3.5]^2}{2} - 30000\frac{[3.5-2]^2}{2} - 40000\frac{[3.5-4.5]^2}{2} - 187400$$

The last bracket is negative so ignore by putting in zero

$$200x10^6\frac{dy}{dx} = 35710\frac{[3.5]^2}{2} - 30000\frac{[3.5-2]^2}{2} - 40000\frac{[0]^2}{2} - 187400$$

$$200x10^6\frac{dy}{dx} = 218724 - 33750 - 187400 = -2426$$

$$\frac{dy}{dx} = \frac{-2426}{200x10^6} - 0.00001213 \text{ and this is the slope at the middle.}$$

$$EIy = 35710\frac{[3.5]^3}{6} - 30000\frac{[3.5-2]^3}{6} - 40000\frac{[3.5-4.5]^3}{6} - 187400[3.5]$$

$$200x10^6\,y = 255178 - 16875 - 0 - 655900 = -417598$$

$$y = \frac{-417598}{200x10^6} - 0.00209 \text{ m or } 2.09 \text{ mm}$$

WORKED EXAMPLE No.7

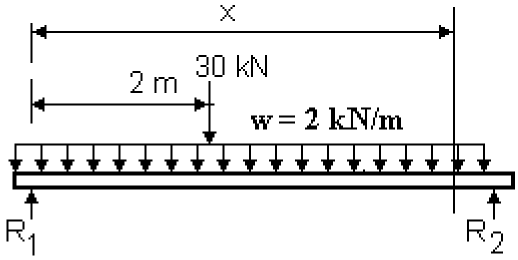

Figure 8

The beam shown is 6 m long with an E I value of 300 MNm^2. Determine the slope at the left end and the deflection at the middle.

SOLUTION

First solve the reactions by taking moments about the right end.
$30 \times 4 + 2 \times 6^2/2 = 6 R_1 = 156$ hence $R_1 = 26$ kN
Total downwards load is 30 + (6 x 2) = 42 kN
$R_2 = 42 - 26 = 16$ kN
Next write out the bending equation.

$$EI\frac{d^2y}{dx^2} = M = R_1[x] - 30000[x-2] - \frac{wx^2}{2}$$

$$EI\frac{d^2y}{dx^2} = 26000[x] - 30000[x-2] - \frac{2000x^2}{2}$$

Integrate once treating the square bracket as the variable.

$$EI\frac{dy}{dx} = 26000\frac{[x]^2}{2} - 30000\frac{[x-2]^2}{2} - \frac{2000[x]^3}{6} + A \ldots\ldots(1)$$

Integrate again

$$EIy = 26000\frac{[x]^3}{6} - 30000\frac{[x-2]^3}{6} - \frac{2000[x]^4}{24} + Ax + B \ldots\ldots(2)$$

BOUNDARY CONDITIONS
x = 0, y = 0   and x = 6 y = 0

Using equation 2 and putting x = 0 and y = 0 we get

$$EI(0) = 26000\frac{[0]^3}{6} - 30000\frac{[0-2]^3}{6} - \frac{2000[0]^4}{24} + A(0) + B$$

Ignore any bracket containing a negative value.

$$0 = 0 - 0 - 0 + 0 + B \quad \text{hence } B = 0$$

Using equation 2 again but this time x = 6 and y = 0

$$EI(0) = 26000\frac{[6]^3}{6} - 30000\frac{[6-2]^3}{6} - \frac{2000[6]^4}{24} + A(6) + 0$$

$$EI(0) = 936000 - 320000 - 108000 + (6) + 6A$$

$$6A = -508000$$

$$A = -84557$$

Now use equations 1 with x = 0 to find the slope at the left end.

$$EI\frac{dy}{dx} = 260000\frac{[0]^2}{2} - 30000\frac{[0-2]^2}{2} - 2000\frac{[0]^3}{6} - 84557$$

Negative brackets are made zero

$$300\ x10^6\frac{dy}{dx} = -84557$$

$$\frac{dy}{dx} = \frac{-84557}{300x10^6} - 0.000282 \text{ and this is the slope at the left end.}$$

Now use equations 2 with x = 3 to find the deflection at the middle.

$$EIy = 26000\frac{[3]^3}{6} - 30000\frac{[3.5-2]^3}{6} - \frac{2000[3]^4}{24} - 84557[3]$$

$$300x10^6\ y = 117000 - 16875 - 6750 - 253671 = -160296$$

$$y = \frac{-160296}{300x10^6} - 0.000534 \text{ m or } 0.534 \text{ mm}$$

SELF ASSESSMENT EXERCISE No.5

1. Find the deflection at the centre of the beam shown. The flexural stiffness is 20 MNm^2. **(0.064 mm)**

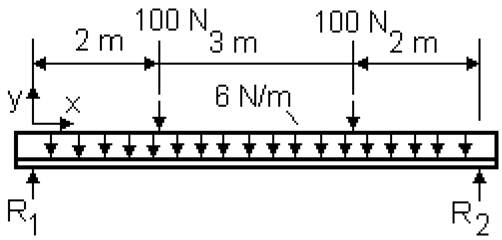

Figure 9

2. Find the deflection of the beam shown at the centre position. The flexural stiffness is 18 MNm^2. **(1.6 mm)**

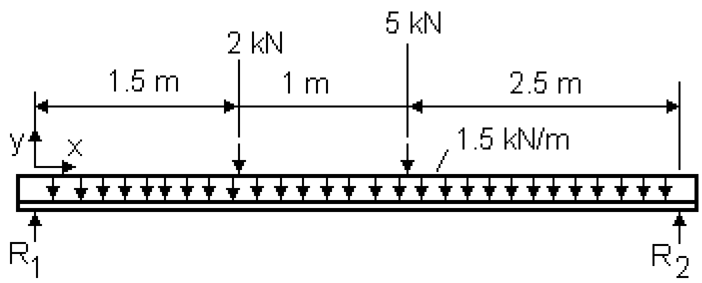

Figure 10

3. Find value of E I which limits the deflection of the beam shown at the end to 2 mm. **(901800 Nm^2)**

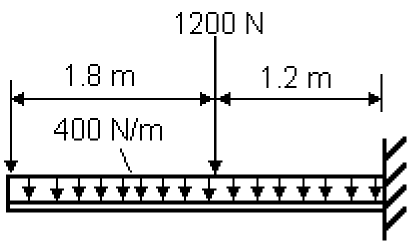

Figure 11

4. A cantilever is 5m long and has a flexural stiffness of 25 MNm^2. It carries a point load of 1.5 kN at the free end and a u.d.l. of 500 N/m along its entire length. Calculate the deflection and slope at the free end.
(-4.06 mm and 1.167 x 10^{-3})

5. A cantilever beam is 6 m long and has a point load of 800 N at the free end and a u.d.l. of 400 N/m along its entire length. Calculate the flexural stiffness if the deflection is 1.5 mm downwards at the free end.
(81.6 MNm^2).

6. A simply supported beam is 6 m long and has a flexural stiffness of 3 MNm^2. It carries a point load of 800 N at the middle and a u.d.l. of 400 N/m along its entire length. Calculate the slope at the ends and the deflection at the middle.
   (1.8 x 10^{-3} and 3.45 mm).

7. Calculate the flexural stiffness of a simply supported beam which will limit the deflection to 2 mm at the middle. The beam is 5 m long and has a point load of 1.2 kN at the middle and a u.d.l. of 600 N/m along its entire length.
   (4 MNm^2).

   The beam has a solid rectangular section twice as deep as it is wide. Given the modulus of elasticity is 120 GPa, calculate the dimensions of the section.
   (168 mm x 84 mm).

8. ENCASTRÉ BEAMS

An encastré beam is one that is built in at both ends. As with a cantilever, there must be a bending moment and reaction force at the wall. In this analysis it is assumed that

- there is no deflection at the ends.
- the ends are horizontal.
- the beam is free to move horizontally.

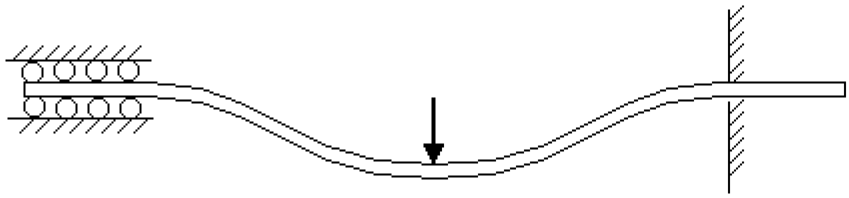

Figure 12

First let us consider two standard cases, one with a point load at the middle and one with a uniformly distributed load. In both cases there will be a reaction force and a fixing moment at both ends. We shall use Macaulay's method to solve the slope and deflection.

8.1 POINT LOAD

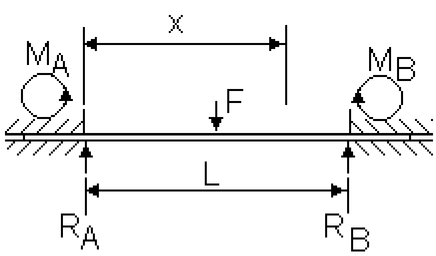

Figure 13

In this case $R_A = R_B = F/2$

The bending moment at distance x from the left end is :

$$M = EI\frac{d^2y}{dx^2} = R_A x - F\left[x - \frac{L}{2}\right] + M_A$$

$$EI\frac{d^2y}{dx^2} = F\frac{x}{2} - F\left[x - \frac{L}{2}\right] + M_A$$

Integrate $$EI\frac{dy}{dx} = F\frac{x^2}{4} - \frac{F\left[x - \frac{L}{2}\right]^2}{2} + M_A x + A$$

Since the slope is zero at both ends it then putting $\frac{dy}{dx} = 0$ and $x = 0$ yields that $A = 0$

$$EI\frac{dy}{dx} = F\frac{x^2}{4} - \frac{F\left[x - \frac{L}{2}\right]^2}{2} + M_A x \ldots\ldots\ldots\ldots(1)$$

Integrate again $$EIy = F\frac{x^3}{12} - \frac{F\left[x - \frac{L}{2}\right]^3}{6} + M_A\frac{x^2}{2} + B$$

Since the deflection is zero at both ends then putting $y = 0$ and $x = 0$ yields that $B = 0$

$$EIy = F\frac{x^3}{12} - \frac{F\left[x - \frac{L}{2}\right]^3}{6} + M_A\frac{x^2}{2} \ldots\ldots\ldots\ldots(2)$$

The constants of integration A and B are always zero for an encastré beam but the problem is not made easy because we now have to find the fixing moment M.

Equations 1 and 2 give the slope and deflection. Before they can be solved, the fixing moment must be found by using another boundary condition. Remember the slope and deflection are both zero at both ends of the beam so we have two more boundary conditions to use. A suitable condition is that y = 0 at x = L. From equation 2 this yields

$$EI(0) = \frac{FL^3}{12} - \frac{F\left[L - \frac{L}{2}\right]^3}{6} + \frac{M_A L^2}{2}$$

$$0 = \frac{FL^3}{12} - \frac{F\left[\frac{L}{2}\right]^3}{6} + \frac{M_A L^2}{2}$$

$$0 = \frac{FL^3}{12} - \frac{FL^3}{48} + \frac{M_A L^2}{2}$$

$$0 = \frac{3FL^3}{148} + \frac{M_A L^2}{2}$$

$$\frac{M_A L^2}{2} = -\frac{3FL^3}{148}$$

$$M_A = -\frac{FL}{8}$$

If we substitute x = L/2 and M_A = -FL/8 the slope and deflection at the middle from equations 1 and 2 becomes :

$$\frac{dy}{dx} = 0 \qquad y = -\frac{FL^3}{192EI}$$

8.2 UNIFORMLY DISTRIBUTED LOAD.

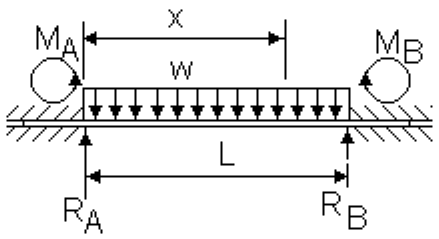

Figure 14

In this case $R_A = R_B = wL/2$

The bending moment at distance x from the left end is :

$$M = EI\frac{d^2y}{dx^2} = R_A x - \frac{wx^2}{2} + M_A$$

$$EI\frac{d^2y}{dx^2} = \frac{wLx}{2} - \frac{wx^2}{2} + M_A$$

Integrate $\quad EI\frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} + M_A x + A$

Since the slope is zero at both ends it then putting $\frac{dy}{dx} = 0$ and $x = 0$ yields that $A = 0$

$$EI\frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} + M_A x \text{..............(1)}$$

Integrate again $\quad EIy = \frac{wLx^3}{12} - \frac{wx^4}{24} + \frac{M_A x^2}{2} + B$

Since the deflection is zero at both ends then putting $y = 0$ and $x = 0$ yields that $B = 0$

$$EIy = \frac{wLx^3}{12} - \frac{wx^4}{24} + \frac{M_A x^2}{2} \text{..............(2)}$$

As in the other case, A and B are zero but we must find the fixing moment by using the other boundary condition of y = 0 when x = L

$$EI(0) = \frac{wL^4}{12} - \frac{wL^4}{24} + \frac{M_A L^2}{2}$$

$$0 = \frac{wL^4}{24} + \frac{M_A L^2}{2}$$

$$M_A = -\frac{wL^2}{12}$$

If we substitute x = L/2 and $M_A = -wL^2/12$ into equations 1 and 2 we get the slope and deflection at the middle to be

$$\frac{dy}{dx} = 0 \text{ and } y = -\frac{wL^4}{384EI}$$

The same approach may be used when there is a combination of point and uniform loads.

SELF ASSESSMENT EXERCISE No. 6

1. Solve the value of EI which limits the deflection under the load to 0.05 mm. **(Ans. 1.53 GNm^2)**

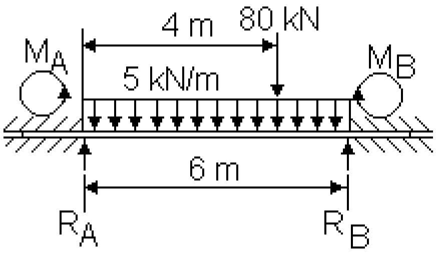

Figure 15

2. Solve the value of EI which limits the deflection under at the middle to 0.2 mm. **(Ans. 11 MNm^2)**

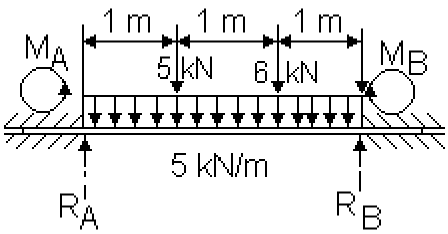

Figure 16